SUR LE CALCUL DES PROBABILITÉS*

Mr. J. A. Mallet
Professor of Astronomy at Geneva

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1 ° On a case of the Ars Conjectandi
of Mr. Bernoulli.

The celebrated Jacques Bernoulli gives in his work *Ars Conjectandi*, page. 161 the solution of the following Problem. Two persons A & B play together with a single die, & agree that each recast the die as many times as it has brought forth points on the first cast, that the one here will win a sum 1, who will bring forth the most points in all his throws, & that if both obtain the same number of points, they will divide equally the proposed sum, but soon one of the players B, bored of the game, offers, instead of throwing the dice, to take 12 points for his part, A consents, we ask who has the greatest expectation to win?[1]

The method which seems most simple to resolve this question, is to seek, if the number of points, which A can attain in all his casts, is greater or lesser than 12, now we find it equal to $12\frac{1}{4}$: It seems therefore, that since the expectation to win depends on the number of points that one expects, we must conclude that the expectation of A is greater than that of B, however if we seek this expectation exactly, we find it smaller than that of B, in the ratio of 15295 to 15809. *Quamquam profecto* (says Mr. Bernoulli) *difficile dictu est, cur ille plura quam hic puncta, minorem autem depositi partem expectet, cum tamen acquisitio depositi, vi pacti, pendeat a punctorum pluralitate*.[2]

We can see by this kind of paradox, how it is easy to be deceived in the solution of these questions, & how it is necessary to use precautions in the reasonings which we make on this matter. Mr. Bernoulli being content to indicate this apparent singularity, without giving the explication of it, I have believed that it would not be useless to enter into greater detail on this subject, in order to clarify perfectly this little difficulty, we will see that we can imagine an infinity of cases similar to that of Mr. Bernoulli, in the solution in the solution of which it would be also easy to be induced to error.

The number of points which a player can expect to obtain in a certain game, is the number which results from the sum of the products of each number of points which he

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*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. November 28, 2009

[1] *Translator's note*: This is the same problem studied by Fuss.

[2] It is difficult to see why the former expects more points but less of the stake, when acquisition of the stake depends on the number of points.

can bring forth, multiplied by the probability that he will obtain this number of points. Thus, for example, in a game where of four cases which can happen equally, one of them obtains 1 point, the second obtains 2 points, & each of the two other, 3 points, the number of points to expect is in this case

$$= \frac{1}{4} \times 1 + \frac{1}{4} \times 2 + \frac{2}{4} \times 3 = 2\frac{1}{4}.$$

This being put, I will remark first, that if each point moreover contributed to increase the gain of the player A, then his expectation would depend well on the number of points which he attains, & if this number would be greater than that which B has taken, which incurs the opportunity of chance not at all, we could conclude with reason that the expectation of A would be greater than that of B. But in the Problem, of which there is concern, this is not at all the same case, because if the game gives to A more than 12 points, he will have won, whatever be the number of points which he obtains above 12, likewise he will have lost, if he obtains any number below it, the question is only to know, if he will pass or if he will not pass the number 12, little import if he surpass it by little or by much. It seems therefore that the ratio of the expectations of the 2 players, can not be concluded from that if the fixed number of points, taken by one of the players, is found greater or lesser, than the number which chance promises, but that it depends on the nature of the game, & that we must examine necessarily in detail all the possible cases.

We suppose generally any game between A & B, where the one will have obtained the most points, will win the stake of the other, without having regard to the surplus of the number of points, which he will have over his adversary, B takes the fixed number of points t, without exposing himself to chance, & A by the nature of the game attains a number s, we see if from the ratio of t to s, it is possible to conclude, which of the two has the greater expectation to win.

There can happen three different cases.

1° If A obtains a number of points t equal to that of B, that which will give to him the sum $\frac{1}{2}$.

2° If he obtains more than t points, that which will give to him the entire sum 1.

3° If he has less than t points, that which will make him lose, or will give to him 0.

Therefore the expectation of A is equal to the expectation of bringing forth t points $+\frac{1}{2}\times$ the expectation of bringing forth precisely t points.

Let this game be such that the expectation

to obtain precisely	1	point is	$= \alpha$	& to obtain more than	0	points is	$= a$
	2		$= \beta$		1	$= b$	
	3		$= \gamma$		2	$= c$	
	$\vdots$		$\vdots$		$\vdots$	$\vdots$	
	t		$= \mu$		$t-1$	$= m$	
	$t+1$		$= \nu$		t	$= n$	
	$\vdots$				$\vdots$		
	u	points	$= \zeta$		$u-1$	points	$= Z$

Then the expectation of A will be equal to $n + \frac{1}{2}\mu$.

Now by the nature of the sequences α, β, *&c.* $\ldots a$, b, *&c.* it is evident that we have $a = \alpha + \beta + \gamma \ldots + \zeta$ & $b = \beta + \gamma \ldots + \zeta$ & in general the x^{th} term of the sequence $a, b, c,$ *&c.* equal to the sum of the x, $x+1$ &c. terms of the sequence α, β *&c.* Therefore we will have $\alpha = a - b$, $\beta = b - c$ *&c.* & $\mu = m - n$. In such a way that the expectation of A will be $\frac{m+a}{2}$ and it will be smaller, greater, or equal to that of B, according as $m + n$ will be smaller, greater, or equal to unity, without any regard to the number s of points which A expects.

It is true that this number s depends on the sequences α, β *&c.* a, b, *&c.* because we have $s = \alpha + 2\beta + 3\gamma \ldots + u\zeta = a + b + c + \&c. \ldots + Z$, but it is easy to see that this sequence a, b, c, *&c.* depending on the nature of the game, can vary to infinity, without the sum s changing in value, that which will give together as many different values in the quantity $m - n$.

Whence it follows evidently, that from the ratio of t to s, we can conclude nothing on the expectations of A & of B.

We will clarify this by some examples, where we will suppose $t = s$.

1° A casts two ordinary dice, we know that he can expect 7 points from this game here. B takes also 7 points for his part, we have therefore $s = t = 7$.

& the Sequence a, b, *&c.* becomes

$$\begin{array}{l}1 \quad\; 2 \quad\; 3 \quad\; 4 \quad\; 5 \quad\; 6 \quad\; 7 \quad\; 8\\ \frac{36}{36}+\frac{36}{36}+\frac{35}{36}+\frac{33}{36}+\frac{30}{36}+\frac{26}{36}+\frac{21}{36}+\frac{15}{36}+\frac{10}{36}+\frac{6}{36}+\frac{3}{36}+\frac{1}{36}=7\end{array}$$

the quantity $\frac{m+n}{2}$ which expresses the expectation of A is $= \frac{\frac{21}{36}+\frac{15}{36}}{2} = \frac{1}{2}$ equal to that of B.

We suppose 2° that A draws from a sack where there are 6 Tickets, one marked 3 points, the other 5, the other 7, another 8, another 9, & another 12, our Sequence will be

$$\begin{array}{l}1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 6 \;\; 7 \;\; 8\\ \frac{6}{6}+\frac{6}{6}+\frac{6}{6}+\frac{6}{6}+\frac{5}{6}+\frac{4}{6}+\frac{4}{6}+\frac{3}{6}+\frac{2}{6}+\frac{1}{6}=7\end{array}$$

& the expectation of $A = \frac{\frac{4}{6}+\frac{3}{6}}{2} = \frac{7}{12}$ greater than that of B.

Finally 3° if A draws from a sack where there are six tickets, one of 4, two of 6, another of 7, one of 9 & one of 10, the Sequence will be

$$\begin{array}{l}1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 6 \;\; 7 \;\; 8\\ \frac{6}{6}+\frac{6}{6}+\frac{6}{6}+\frac{5}{6}+\frac{5}{6}+\frac{5}{6}+\frac{3}{6}+\frac{2}{6}+\frac{2}{6}+\frac{1}{6}=7\end{array}$$

the expectation of $A = \frac{5}{12}$ smaller than that of B.

Here are therefore three cases where $s = t$, & where however the expectation of A is found in one equal to that of B, in the 2nd greater, & in the 3rd smaller.

We can likewise imagine an infinity of games, where t being smaller than s the expectation of B, is however greater than that of A. Here is an example of it drawn from the Probabilities of human life.

An infant who comes to be born has according to the mortality Tables of Halley the expectation around 27 years of life; A proposes to wager that this infant will not attain his 12th year; there are few people who would not accept gladly this wager, as very advantageous, however we find by these same Tables, that of 1300 newborns there are 654 dead at the end of 12 years, so that the expectation of A is found to be $= \frac{654}{1300}$, that is greater than that of his adversary.

The same Sequence can be such that its sum s is infinitely greater than t, & however the expectation of A smaller than that of B.

We will find entirely also easily some cases where the expectation of B is smaller than that of A, whenever t is greater than s. Let for example a die of 12 faces of which four are marked 1, one 4, one 9, five others 11, & the twelfth 12. The number s of points to expect for A is 7, it is necessary that B take at least $t = 10$, in order that he has some advantage over A.

We can propose here a question to resolve, namely the excess of t over s being determined, & suppose also as great as we will wish, is it possible that there are some cases where the expectation of A is nonetheless greater than that of B? or else is there a certain ratio of t to s beyond which it is impossible that the expectation of A surpasses that of B?

I will remark first, that by supposing t infinitely greater than s, it is impossible that the expectation of A surpass that of B, because it would be necessary for this that the Sequence a, b, *&c.* having an infinite number of terms, has however a finite sum, that which is impossible in this case here, where the denominator of these fractions is everywhere the same, it would be necessary moreover that the sum of two terms the t^{th} & $\overline{t+1}^{\text{st}}$ taken to infinity were greater than unity, that which can not be, if s is finite, because s is always greater than t times the t^{th} term.

We seek therefore for a determined value of s the greatest value possible of t which renders $m + n$ greater than unity. Let for example $s = 5$ & the Series

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ \frac{12}{12}+ & \frac{7}{12}+ & \frac{7}{12}+ & \frac{7}{12}+ & \frac{7}{12}+ & \frac{7}{12}+ & \frac{7}{12}+ & \frac{6}{12}=5 \end{array}$$

the greatest value of t ($\overline{m+n}$ remaining 71) will be $= 7$; this Series arranged in every other manner, will give a value of t less than 7.

In general the denominator being n, the Series arranged the best way in order to render t the greatest possible will be

$$\frac{n}{n} + \frac{\frac{1}{2}n+1}{n} + \frac{\frac{1}{2}n+1}{n} + \&c. \cdots + \frac{\frac{1}{2}n+1}{n} + \frac{\phi}{n} = s$$

($\frac{\phi}{n}$ is the last term which is smaller than $\frac{\frac{1}{2}n+1}{n}$ & which can be zero in certain cases.)

Therefore the greatest value of t will be equal to the quotient of the division of $s - 1 - \frac{\phi}{n}$ by $\frac{\frac{1}{2}n+1}{n}$ that is to say

$$t = \frac{\overline{s-1} \cdot n}{\frac{1}{2}n+1} - \frac{\phi}{\frac{1}{2}n+1} = 2 \times \overline{s-1} - \frac{4 \cdot \overline{s-1} + 2\phi}{n+2}$$

a value which will be so much greater as the number n of all the possible cases will be greater, & which becomes $= 2 \times \overline{s-1}$ when this number n will be infinity.

It is therefore impossible that the expectation of A, who awaits a certain number of points s be greater than that of B, if this B has already more than $2 \times \overline{s-1}$ points.

Thus for example in a Lottery of which the ticket is worth 10 écus, we can wager not at all that this ticket will return at least a certain sum, if we do not know the manner in which the lots are distributed, & in all the ordinary Lotteries, there is always a very great probability that a ticket will return nothing; but it is possible that a Lottery is arranged in a fashion that we can wager without disadvantage, that a single ticket of 10 écus will return more than 19, this is that which would happen for example if all the lots were equal & doubles of the wager, but there is no other combination which can make wagering that a single ticket will return more than double its value.

2° CALCULATION of a particular LOTTERY.

A Particular was once in Paris a rather singular Lottery, of which here are the Conditions

1° The number of tickets were one million or 1000000, each cost 10 sols, this which returned 500000 Livres. Of this Sum it retained 75000 Livres or 15 percent, there remained 425000 Livres to distribute in 20000 lots.

2° One put all the Tickets into a box, & all the lots into another, after which one drew one Ticket and one lot, one replaced next this same Ticket into the box, so that the same Ticket could have many lots, & even have all of them; namely if it were drawn twenty thousand times in sequence.

3° If someone had 50 tickets, one returned to him his 50 stakes or 25 Livres if none of his 50 Tickets came out with a lot, thus the Banker or the Master of the Lottery was himself exposed to the condition of fortune, & could lose or win by his Lottery.

We suppose that there are only 20000 persons who have put into this Lottery, that is to say that each has taken 50 tickets, that which will be probably happening in order to be able to enjoy the benefit in article 3.

The case most favorable to the Banker, is that each person is able to grab a lot with his half a hundred; he has thus 75000 Livre profit which he has first levied on the total.

The most disadvantageous case, is that the same person has all the lots, there will remain yet 19999 persons to each of whom, it is necessary to give 25 Livres that which makes 499975 Livres so that in this case the Banker would lose 424975 Livres.

We can see also that if the 20000 lots were distributed among 17000 persons, he would be obliged then to compensate 3000 of them, that which would make 75000 Livres so that in this case he would have neither gain nor loss.

But if we wish to seek exactly, what is the expectation of the Banker, we suppose in order to resolve this Problem generally, that here are n players, who each take an equal number of tickets, for which each pays a sum b, which will be rendered to him, if none of the tickets has some lot, & let a be the sum that the Banker levies first on the total of the wagers.

We will seek first the probability that each player will have a lot,

II° The probability that any single player will have none of the lots at all, in which case the Banker will render the sum b.

III° The probability that two players only will have none of the Lots at all, & in this case the Banker renders the sum $2b$, & thus in sequence to the end, namely when a single player will have all the lots, & that the Banker will render the sum $(n-1)b$.

In order to find the probability in the I$^{\text{st}}$ case that each player will have a lot.

We suppose 1° that $\overline{n-1}$ players have already had $\overline{n-1}$ lots; what is the probability that the lot which remains, will go to the one who has had nothing yet? It is evident that it is $= \frac{1}{n}$.

2° If $n-2$ players have already had $\overline{n-2}$ lots, what is the probability that each of the two others who have had nothing yet, will have a lot? We see that there are $\overline{n-2}$ cases in order that this not happen, & two cases in order to fall into the preceding case, that is to say in order to have the probability $\frac{2}{n}$. Therefore the probability sought is $\frac{\overline{n-2}\times 0+2\times\frac{1}{n}}{n} = \frac{1.2}{n^2}$.

3° If $\overline{n-3}$ players have had $\overline{n-3}$ lots, in order to find the probability that each of the three others will have one of them, there are $\overline{n-3}$ cases in order that this not happen, & 3 cases in order to fall into the preceding case, that which gives the probability sought $= \frac{1.2.3}{n^3}$.

It is easy to see after this that if $\overline{n-p}$ players have already had $\overline{n-p}$ lots, the probability that each of the others will have one of them is $= \frac{1.2.3\ldots p}{n^p}$.

And if there is yet any ticket to draw, the probability that each player will have a lot, is found $= \frac{1.2.3\ldots n}{n^n}$ (I^{st} case)

We follow this same method in order to seek the probability of the II^{nd} case, that a single player will have none of the lots at all.

1° If $\overline{n-1}$ players have had $\overline{n-1}$ lots; what is the probability that the lot which remains will not go to the one who has had nothing yet? There are $\overline{n-1}$ favorable cases & 1 to the contrary, that which gives this probability $= \frac{n-1}{n}$.

2° If $\overline{n-2}$ players have had $\overline{n-2}$ lots, what is the probability that one alone of the two lots will go to one of the two players who have had nothing?

There are two cases to fall into the preceding cases, & $\overline{n-2}$ cases to have the expectation $\frac{2}{n}$ that if $\overline{n-2}$ players have already $\overline{n-1}$ lots, the lot which remains will go to one of the two players who have nothing, so that the probability sought is

$$= \frac{2\times\frac{n-1}{n}+\overline{n-2}\times\frac{2}{n}}{n} = 1.2.\left(\frac{\overline{n-1}+\overline{n-2}}{n^2}\right).$$

The following case is resolved in the same manner, & we will find the probability

$$= 1.2.3.\left(\frac{\overline{n-1}+\overline{n-2}+\overline{n-3}}{n^3}\right).$$

Whence we can conclude that if $\overline{n-p}$ players have had $\overline{n-p}$ lots, the probability that one alone of the p remaining will have nothing is

$$= 1.2.3\ldots p\left(\frac{\overline{n-1}+\overline{n-2}+\cdots+\overline{n-p}}{n^p}\right)$$

Therefore the probability before the drawing, that one alone will have none of the lot at all is

$$= 1.2.3\ldots n\left(\frac{1+2+3\cdots+\overline{n-1}}{n^n}\right)\ (\mathrm{II}^{\text{nd}}\text{ case})$$

By following the same method, we will find that the probability that two players only will have none of the lot at all will be

$$= \frac{3.4.5\ldots n}{n^n}\left(1^2+2^2+3^2 \text{ \&c.} + \overline{n-2}^2 + 1.2+1.3+1.4 \text{ \&c. } + 1.\overline{n-2}+2.3+2.4 \text{ \&c.}\right)$$

(III$^{\text{rd}}$ case)

The second member contained within the parentheses, is the sum of the products two by two of the first $\overline{n-2}$ natural numbers, or that which is the same thing, the square of the polynomial composed of the $\overline{n-2}$ natural numbers by making all the coefficients $= 1$.

We will find likewise that the probability that only three players will have none of the lot at all will be $= \frac{4.5\ldots n}{n^n}$ multiplied by all the products three by three, which we can make with the first $\overline{n-3}$ natural numbers, I will call this second factor, the cube without coefficients of the first $\overline{n-3}$ numbers, & I will designate thus the sought probability

$$\frac{4.5\ldots n}{n^n}\,(1+2+3\cdots+n-3)^{3\text{rd}} \text{ (IV}^{\text{th}}\text{ case)}$$

Here is the recapitulation of these different cases:

$$\begin{array}{ll}
\text{I}^{\text{st}}\text{ case the Probability} & =\frac{1.2.3\ldots n}{n^n}\,(1+2+3+\text{\&c. }\cdots+n)^{0^{\text{th}}} \\
\text{II}^{\text{nd}}\text{ case} & =\frac{2.3.4\ldots n}{n^n}\,(1+2+3+\text{\&c. }\cdots+n-1)^{1^{\text{st}}} \\
\text{III}^{\text{rd}}\text{ case} & =\frac{3.4.5\ldots n}{n^n}\,(1+2+3+\text{\&c. }\cdots+n-2)^{2^{\text{nd}}} \\
\quad\vdots & = \quad\vdots \\
\text{P}^{\text{th}}\text{ case} & =\frac{p.p+1.p+2\ldots n}{n^n}\,(1+2+3+\text{\&c. }\cdots+n-p+1)^{\overline{p-1}^{\text{th}}}
\end{array}$$

We see presently what will be the expectation of the Banker.

He has the sum a of which he renders nothing if all have some lots; of which he renders b, if $\overline{n-1}$ players have lots; of which he renders $2b$, if $\overline{n-2}$ players have lots &c. So that his expectation will be

$$\begin{array}{l}
(a)\,\frac{1.2.3\ldots n}{n^n}\,(1+2+3+\&c.\cdots+n)^{0^{\text{th}}} \\
+(a-b)\,\frac{2.3\ldots n}{n^n}\,\left(1+2+3+\&c.\cdots+\overline{n-1}\right)^{1^{\text{st}}} \\
+(a-2b)\,\frac{3.4\ldots n}{n^n}\,\left(1+2+3+\&c.\cdots+\overline{n-2}\right)^{2^{\text{nd}}} \\
\quad\&c. \\
+(a-\overline{n-2}.b)\,\frac{\overline{n-1}.n}{n^n}(1+2)^{\overline{n-2}^{\text{nd}}} \\
+(a-\overline{n-1}.b)\,\frac{n}{n^n}(1)^{\overline{n-1}^{\text{st}}}
\end{array}$$

Let the Factor $\frac{n}{n^n}(1)^{\overline{n-1}^{\text{st}}}$ which multiplies $a-\overline{n-1}.b$ be equal to A, that which multiplies $a-\overline{n-2}.b$ be equal to B, the following to C, D &c. that which multiplies $a-2b$ be equal to P, the following Q & finally the last $= R$, so that the expectation of the Banker is

$$(a-\overline{n-1}.b)A + (a-\overline{n-2}.b)B\ \&c.\cdots + (a-2b)P + (a-b)Q + (a)R$$

The coefficient of a will be therefore $= A + B + C + \&c. \cdots + Q + R$ which is evidently equal to unity, since it expresses the sum of the Probabilities of different cases, among which there is certainly one which will take place.

The coefficient of b will be

$$= \overline{n-1}.A + \overline{n-2}.B + \&c. \cdots + 2P + Q.$$

The concern therefore presently is to determine these quantities $A,\ B\ \&c.$

We seek therefore for this purpose what the quantity $(1+2+3+\&c.\ \cdots + p)^{p^{\text{th}}}$ is (I intend always by the p^{th} power of the polynomial putting all the coefficients in it there equal to unity.

We will resolve this Problem generally, by seeking to expand a quantity of this nature $(1+2+3+\&c.\ \cdots + q)^{p^{\text{th}}}$.

1° For two quantities.

We find

$$(a+b)^{0^{\text{th}}} = \frac{b}{b-a} - \frac{a}{b-a}$$
$$(a+b)^{1^{\text{st}}} = \frac{b^2}{b-a} - \frac{a^2}{b-a}$$
$$(a+b)^{2^{\text{nd}}} = aa + ab + bb = \frac{b^3}{b-a} - \frac{a^3}{b-a}$$
$$(a+b)^{3^{\text{rd}}} = a^3 + a^2b + ab^2 + b^3 = \frac{b^4}{b-a} - \frac{a^4}{b-a}$$

& generally

$$(a+b)^{p^{\text{th}}} = \frac{b^{p+1}}{b-a} = \frac{a^{p+1}}{b-a}$$

2° For three quantities

$$\begin{aligned}
(a+b+c)^2 &= (a+b)^2 + (a+b) + cc \\
&+ \frac{b^3-a^3}{b-a} + \frac{b^2-a^2}{b-a} + c\frac{b-a}{b-a}cc \\
&= \frac{1}{b-a}\left\{\begin{array}{l} +b^3 + bbc + bcc \\ -a^3 - a^3c - acc \end{array}\right\} \\
&= \frac{1}{b-a}\left\{\begin{array}{l} +b^3\dfrac{c^3+b^3}{c-b} \\ -a\dfrac{c^3-a^3}{a-c} \end{array}\right\} \\
&= \frac{1}{b-a}\cdot\frac{c^3-b^3}{c-b} - \frac{a}{b-a}\cdot\frac{c^3-a^3}{c-a} \\
&= \frac{c^4}{(c-b)(c-a)} - \frac{b^4}{(c-b)(b-a)} + \frac{a^4}{(c-a)(b-a)}
\end{aligned}$$

By following the same method, we find

$$(a+b+c)^3 = \frac{c^5}{(c-b)(c-a)} - \frac{b^5}{(c-b)(b-a)} + \frac{a^5}{(c-a)(b-a)}$$

& generally

$$(a+b+c)^p = \frac{c^{p+2}}{(c-b)(c-a)} - \frac{b^{p+2}}{(c-b)(b-a)} + \frac{a^{p+2}}{(c-a)(b-a)}$$

3° For four quantities

$$\begin{aligned}
(a+b+c+d)^3 =& (a+b+c)^3 + (a+b+c)^2 d \\
& + (a+b+c)^1 dd + (a+b+c)^0 d^3 \\
=& \frac{c^5}{(c-b)(c-a)} - \frac{b^5}{(c-b)(b-a)} + \frac{a^5}{(c-a)(b-a)} \\
& + \frac{c^4 d}{(c-b)(c-a)} + \frac{b^4 d}{(c-b)(c-a)} - \frac{a^4 d}{(c-a)(b-a)} \\
& + \frac{c^3 dd}{(c-b)(c-a)} - \frac{b^3 dd}{(c-b)(b-a)} + \frac{a^3 dd}{(c-a)(b-a)} \\
& + \frac{c^2 d^3}{(c-b)(c-a)} - \frac{bbd^3}{(c-b)(b-a)} + \frac{aad^3}{(c-a)(b-a)} \\
=& \frac{cc}{(c-b)(c-a)} \cdot \frac{d^4-c^4}{d-c} - \frac{bb}{(c-b)(b-a)} \cdot \frac{d^4-b^4}{d-b} \\
& + \frac{aa}{(c-a)(b-a)} \cdot \frac{d^4-a^4}{d-b} \\
=& \frac{d^6}{(d-c)(d-b)(d-a)} - \frac{c^6}{(d-c)(c-b)(c-a)} \\
& + \frac{b^6}{(d-b)(c-b)(b-a)} - \frac{a^6}{(d-a)(c-a)(b-a)}
\end{aligned}$$

& generally

$$\begin{aligned}
(a+b+c+d)^p =& \frac{d^{p+3}}{(d-c)(d-b)(d-a)} - \frac{c^{p+3}}{(d-c)(c-b)(c-a)} \\
& + \frac{b^{p+3}}{(d-b)(c-b)(b-a)} - \frac{a^{p+3}}{(d-a)(c-a)(b-a)}
\end{aligned}$$

It is not difficult to discover the law for 5, 6 quantities or further, that of the Numerators is evident, & for the Denominators, we will see that in general for $(a+b+ \&c. \cdots + q)^p$ the Denominator of d^{p+f} for example is equal to the Denominator of

d^{p+f-1} (of the preceding case or q it was not) multiplied by $(q-d)$, & that the Denominator of the quantity q^{p+f} is always $(q-a)(q-b)(q-c)$ &c.

We apply now that which we just found to the natural numbers, & we suppose $a=1$, $b=2$, $c=3$, &c.

We will have generally

$$(1+2+3+\&c.\cdots+q)^p = \frac{q^{q+p-1}}{1.2\ldots\overline{q-1}} - \frac{(q-1)^{q+p-1}}{(1.2\ldots\overline{q-2}).1} + \frac{(q-2)^{q+p-1}}{(1.2\ldots\overline{q-3}).1.2}$$
$$-\frac{(q-3)^{q+p-1}}{(1.2\ldots\overline{q-4}).1.2.3} + \frac{(q-4)^{q+p-1}}{(1.2.\ldots\overline{q-5}).1.2.3.4} - \&c.\cdots \pm \frac{(1)^{q+p-1}}{1.2.3\ldots\overline{q-1}}.$$

And giving next to p & q the convenient values we will have

$$A = (1)^{n-1}\frac{n}{n^n}$$
$$B = \left(\frac{2^{n-1}}{1} - \frac{1^{n-1}}{1}\right)\frac{n.\overline{n-1}}{n^n}$$
$$C = \left(\frac{3^{n-1}}{1.2} - \frac{2^{n-1}}{1.1} + \frac{1^{n-1}}{1.2}\right)\frac{n.\overline{n-1}.\overline{n-2}}{n^n}$$
$$D = \left(\frac{4^{n-1}}{1.2.3} - \frac{3^{n-1}}{1.2.1} + \frac{2^{n-1}}{1.1.2} - \frac{1^{n-1}}{1.2.3}\right)\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{n^n}$$
$$\&c$$
$$\vdots$$
$$Q = \left(\frac{\overline{n-1}^{n-1}}{1.2\ldots\overline{n-2}} - \frac{\overline{n-2}^{n-1}}{1.2\ldots\overline{n-3}.1} + \frac{\overline{n-3}^{n-1}}{1.2\ldots\overline{n-4}.1.2} - \&c.\cdots \pm \frac{1^{n-1}}{1.2\ldots\overline{n-2}}\right)\frac{n.\overline{n-1}\ldots3.2}{n^n}$$
$$R = \left(\frac{n^{n-1}}{1.2\ldots\overline{n-1}} - \frac{\overline{n-1}^{n-1}}{1.2\ldots\overline{n-2}.1} + \frac{\overline{n-2}^{n-1}}{1.2\ldots\overline{n-3}.1.2} - \&c.\cdots \mp \frac{1^{n-1}}{1.2\ldots\overline{n-1}}\right)\frac{n.\overline{n-1}\ldots2.1}{n^n}$$

We can arrange these quantities in another manner more accommodating by putting in the same term the n^{n-1}, in another the $\overline{n-1}^{n-1}$ &c. in the following fashion:

$$\alpha = \frac{1.2\ldots n}{1.2\ldots\overline{n-1}} \cdot \frac{n^{n-1}}{n^n}$$

$$\beta = \left(\frac{2.3\ldots n}{1.2\ldots\overline{n-2}} - \frac{1.2\ldots n}{1.2\ldots\overline{n-2}.1}\right) \frac{\overline{n-1}^{n-1}}{n^n}$$

$$=(1-1)\frac{2.3\ldots n}{1.2\ldots\overline{n-2}} \cdot \frac{\overline{n-1}^{n-1}}{n^n}$$

$$\gamma = \left(\frac{3.4\ldots n}{1.2\ldots\overline{n-3}} - \frac{2\ldots n}{1.2\ldots\overline{n-3}.1} + \frac{1.2\ldots n}{1.2\ldots\overline{n-3}.1.2}\right) \frac{\overline{n-2}^{n-1}}{n^n}$$

$$=(1-2+1)\frac{3.4\ldots n}{1.2\ldots\overline{n-3}} \cdot \frac{\overline{n-2}^{n-1}}{n^n}$$

$$\delta = \left(\frac{4.5\ldots n}{1.2\ldots\overline{n-4}} - \frac{3\ldots n}{1.2\ldots\overline{n-4}.1} + \frac{2\ldots n}{1.2\ldots\overline{n-4}.1.2} - \frac{1.2\ldots n}{1.2\ldots\overline{n-4}.1.2.3}\right) \frac{\overline{n-3}^{n-1}}{n^n}$$

$$=(1-3+3-1)\frac{4.5\ldots n}{1.2\ldots\overline{n-4}} \cdot \frac{\overline{n-3}^{n-1}}{n^n}$$

&c.

$$\omega = \left(n - \frac{\overline{n-1}.n}{1} + \frac{\overline{n-2}.\overline{n-1}.n}{1.2} - \&c.\cdots \pm \frac{1.2\ldots n}{1.2\ldots\overline{n-1}}\right) \frac{1^{n-1}}{n^n}$$

We see easily that the sum of all these quantities α, β, &c. is the same as that of the quantities A, B, &c. & we can see also the truth of that which we have said above that this sum which formed the coefficient of a must be equal to unity, because here we have $\alpha = 1$, & all the others α, β, γ, &c. $= 0$.

In order to find the coefficient of b which is $= \overline{n-1}.A + \overline{n-1}.B + \&c.\cdots + 2P + Q$

It is necessary to seek the sum of all the following quantities.

1st	$A + B + C + \&c.\cdots + O + P + Q$
2nd	$A + B + C + \&c.\cdots + O + P$
3rd	$A + B + C + \&c.\cdots + O$
&c.	&c.
$\overline{n-1}$	$A + B$
n	A

We will find the 1st by taking the sum of the quantities α, β, &c. $\cdots\omega$ after having subtracted from each the last term, which is alternatively positive & negative, because these last terms are precisely the value of R.

By canceling out in this last term the quantity α becomes $= 0$

$$\begin{aligned}
\beta &= +1 \cdot \frac{2.3\ldots n}{1.2\ldots\overline{\overline{n-2}}} \cdot \frac{\overline{n-1}^{n-1}}{n^n} = \beta' \\
\gamma &= -1 \cdot \frac{3.4\ldots n}{1.2\ldots\overline{\overline{n-3}}} \cdot \frac{\overline{n-2}^{n-1}}{n^n} = \gamma' \\
\delta &= +1 \cdot \frac{4.5\ldots n}{1.2\ldots\overline{\overline{n-4}}} \cdot \frac{\overline{n-3}^{n-1}}{n^n} = \delta' \\
&\text{\&c.} \\
\omega &= \mp 1 \cdot \frac{n}{1} \cdot \frac{1^{n-1}}{n^n} = \omega'
\end{aligned}$$

In order to have the 3rd sum $A + B+$ &c. $\cdots + P$, it will be necessary to subtract from the 1st the value of Q which is equal to the sum of the penultimate terms of the quantities $\alpha,\ \beta$,&c.

That way the quantities α & β become $= 0$

$$\begin{aligned}
\text{\&} \quad \gamma &= +1 \cdot \frac{3.4\ldots n}{1.2\ldots\overline{\overline{n-3}}} \cdot \frac{\overline{n-2}^{n-1}}{n^n} = \gamma'' \\
\delta &= -2 \cdot \frac{4.5\ldots n}{1.2\ldots\overline{\overline{n-4}}} \cdot \frac{\overline{n-3}^{n-1}}{n^n} = \delta'' \\
\epsilon &= +3 \cdot \frac{5.6\ldots n}{1.2\ldots\overline{\overline{n-5}}} \cdot \frac{\overline{n-4}^{n-1}}{n^n} = \epsilon'' \\
&\text{\&} \\
\text{\&} \quad \omega &= \pm\overline{n-2} \cdot \frac{n}{1} \cdot \frac{1^{n-1}}{n^n} = \omega''
\end{aligned}$$

In order to have the 3rd sum $A + B+$ &c. $\cdots + O$, it is necessary again to subtract the antepenultimate terms & we will have

$$\begin{aligned}
+1 \cdot \frac{4.5\ldots n}{1.2\ldots\overline{\overline{n-4}}} \cdot \frac{\overline{n-3}^{n-1}}{n^n} &= \delta''' \\
-3 \cdot \frac{5.6\ldots n}{1.2\ldots\overline{\overline{n-5}}} \cdot \frac{\overline{n-4}^{n-1}}{n^n} &= \epsilon''' \\
+6 \cdot \frac{6.7\ldots n}{1.2\ldots\overline{\overline{n-6}}} \cdot \frac{\overline{n-5}^{n-1}}{n^n} &= \xi''' \\
&\text{\&c.}
\end{aligned}$$

The coefficients $1 - 3 + 6$ &c. are the triangular numbers, with the signs alternately positive & negative.

We will follow the same method in order to find the 4th Sums & the coefficients will be the pyramidal numbers with the signs also alternately positives & negatives, & thus for the others.

Taking presently all these quantities, the total sum will be

$$= +1\cdot\left\{\frac{2.3\ldots n}{1.2\ldots\overline{n-2}}\cdot\frac{\overline{n-1}^{n-1}}{n^n}\right\}\begin{matrix}-1\\+1\end{matrix}\left\{\frac{3.4\ldots n}{1.2\ldots\overline{n-3}}\cdot\frac{\overline{n-2}^{n-1}}{n^n}\right\}$$

$$\begin{matrix}+1\\-2\\+1\end{matrix}\left\{\frac{4.5\ldots n}{1.2\ldots\overline{n-4}}\cdot\frac{\overline{n-3}^{n-1}}{n^n}\right\}\begin{matrix}-1\\+2\\-3\\+1\end{matrix}\Big\{\ \&c.$$

$$=\frac{2.3\ldots n}{1.2\ldots\overline{n-2}}\cdot\frac{\overline{n-1}^{n-1}}{n^n}\cdot\text{ equal to the coefficient of the quantity } b.$$

Therefore finally, the expectation of the Banker is

$$\begin{aligned}&= a-\frac{2.3\ldots n}{1.2\ldots\overline{n-2}}\cdot\frac{\overline{n-1}^{n-1}}{n^n}\cdot b\\&= a-n\cdot\left(\frac{n-1}{n}\right)^n b.\end{aligned}$$

We suppose that there are nm tickets, or n players who each take a number m of tickets, each ticket costs the sum b, each person will pay the sum mb, & the Banker levies first for himself a percent on the total sum nmb which all the tickets have produced.

By substituting therefore $\frac{a}{100}nmb$ instead of a & mb for b, the expectation of the Banker will be

$$=\frac{a}{100}nmb-n\cdot\left(\frac{n-1}{n}\right)^n mb.$$

It will be null when $a = 100\left(\frac{n-1}{n}\right)^n$, & this value of a becomes the greatest possible namely $= 100$ when n is infinitely great.

Indeed if the Banker levied first 100 percent, that is to say takes all, as then a person can have none of the lot, he is obliged to render all.

In the particular case where $n = 20000$, $b = 10$ sols, $m = 50$, & $a = 15$.

The expectation of the Banker is negative $= -108900$ Livres nearly, that which makes a prodigious disadvantage, there is appearance that the one who made this Lottery has not taken the pain to make all the preceding calculations.

In this case $\left(\frac{n-1}{n}\right)^n = \frac{36778}{100000}$, thus it would have been necessary that the Banker levied first 36.778 percent, finally to have neither advantage nor disadvantage.

We can see easily that the quantity $\left(\frac{n-1}{n}\right)^n$ will increase as n increases, that which consequently will make the quantity $a = 100\left(\frac{n-1}{n}\right)^n$ increase necessary in order that the advantage of the Banker be null. Thus the quantities a, m & b remaining the same, the advantage of the Banker will diminish in measure as the number n of players will increase, that which we can confirm by some examples.

If	$n = 1$	the advantage of the Banker	$= \frac{1}{100} a.mb$	& in order that it be null, there must be	$a = 0$
	$n = 2$		$= \left(\frac{2}{100} a - 0.5\right) mb$		$a = 25$
	$n = 10$		$= \left(\frac{1}{10} a - 3.487\right) mb$		$a = 34.87$
	$n = 100$		$= (a - 36)\, mb$		$a = 36.00$
	$n = 1000$		$= (10a - 367)\, mb$		$a = 36.70$
	$n = 1000000$		$= (10000a - 367647)\, mb$		$a = 36.76$

Whence we see that the Banker will have some disadvantage when he will levy on the total, less than that which is indicated by the values of a of the last column, there is therefore that a single case, namely when $n = 1$ that the Banker can not have some disadvantage, & his advantage will be proportional to the quantity amb.

If for a determined value of a we would wish to seek the number n of players necessary in order that the Banker have no disadvantage, there is only to deduce the value of n from the equation $100 \times \left(\frac{n-1}{n}\right)^n = a$, we have the equation

$$n. \log \frac{n-1}{n} = \log \frac{a}{100}$$

which we can resolve by approximation.

In the case above where $a = 15$, we find n smaller than 2, because by putting it equal to 2 in it, the Banker has yet 5 Livres of disadvantage.

After having resolved this Problem by the preceding method, which is rather long & seeing that the result becomes so simple from it, I have thought that there would be perhaps a more easy way to come to it at the end. This would be to seek the advantage of any player, to multiply it by n, & to change in it the signs in order to have that of the Banker, because it is evidently equal to the sum of the advantages of each player taken negatively.

I will make for that preceding the following Lemma which could serve for other similar cases.

A die with n faces being cast p times, we ask what is the probability to bring forth precisely m times one of the faces a.

The probability of having a one time in	1 trial is	$= \frac{1}{n}$	$= 1 \times \frac{\overline{n-1}^0}{n}$
	2 trials	$= \frac{1 \cdot \frac{\overline{n-1}}{n} + \overline{n-1} \times \frac{1}{n}}{n}$	$= 2 \times \frac{\overline{n-1}^1}{n^2}$
The probability of having a 1 time in	3 trials is	$= \frac{1 \cdot \frac{\overline{n-1}^2}{n^2} + \overline{n-1} \times \frac{\overline{n-1}}{n^2}}{n}$	$= 3 \times \frac{\overline{n-1}^2}{n^3}$
one time in	p trials		$= p \times \frac{\overline{n-1}^{p-1}}{n^p}$
The probability of having a two times in	2 trials	$= \frac{1}{n^2}$	$= 1 \times \frac{\overline{n-1}^0}{n^2}$
	3 trials	$= \frac{1 \cdot 2 \cdot \frac{\overline{n-1}}{n^2} + \overline{n-1} \times \frac{\overline{n-1}^0}{n^2}}{n}$	$= 3 \times \frac{\overline{n-1}^1}{n^3}$
	4 trials	$= \frac{1 \cdot 3 \cdot \frac{\overline{n-1}^2}{n^3} + \overline{n-1} \cdot 3 \cdot \frac{\overline{n-1}^1}{n^3}}{n}$	$= 6 \times \frac{\overline{n-1}^2}{n^4}$
two times in	p trials		$= \frac{p\,\overline{p-1}}{2} \cdot \frac{\overline{n-1}^{p-2}}{n^p}$

It is not necessary to go further in order to see the law which they follow in the following cases, each probability being deduced from the preceding, we will find thence that the sought probability of bringing forth a, m times, in p trials is

$$= \frac{p.\overline{p-1}.\overline{p-2}\cdots\overline{p-m+1}}{1.2.3\cdots m} \times \frac{\overline{n-1}^{p-m}}{n^p}.$$

We will deduce thence quite easily, the expectation of each person who puts into the Lottery, namely the probability that he will have a lot, the probability that he will have two of them, three &c. because each face of the die can represent a player, & as each cast of the die brings forth a face, likewise each drawing makes the No. of a player exit, in order to have a lot, all things being therefore perfectly similar in these two cases, the formula which we have found will express the expectation which a player has to obtain m lots among the number p which we will deduce from it, & making $p = n$, we will have the expectation to have m lots in the Lottery

$$= \frac{n.\overline{n-1}.\overline{n-2}\cdots\overline{n-m+1}}{1.2.3\cdots m} \cdot \frac{\overline{n-1}^{n-m}}{n^n}$$

Let the 1st lot be α, the 2nd $= \beta$, the 3rd $= \gamma$ &c. & the sum of all these lots $= A$, we know not at all these quantities α, β, γ &c, there is need only to know their sum A, because the expectation of the Banker depends not at all on the manner in which the lots are distributed, but only on their number which equals the one of the players, the expectation of each of these will not depend on it any longer.

Thus the value of a single lot will be expressed by the sum of all the lots divided by n, namely $= \frac{A}{n}$.

The value of two lots will be expressed by the sum of all the lots taken two by two, divided by the number of combinations of n things two by two, that is to say

$$= \frac{(\alpha+\beta)+(\alpha+\gamma)+(\alpha+\delta)\,\&\text{c.}+(\beta+\gamma)+(\beta+\delta)\,\&\text{c}}{\frac{n\times\overline{n-1}}{2}}$$

$$= \frac{\overline{n-1}.A}{\frac{n\times\overline{n-1}}{2}} = \frac{2A}{n}.$$

The value of three lots will be

$$= \frac{\frac{\overline{n-1}.\overline{n-2}}{2}\times A}{\frac{n.\overline{n-1}.\overline{n-2}}{2.3}} = \frac{3A}{n}$$

& in general the value of a number q of lots will be expressed by $\frac{qA}{n}$.

Therefore the expectation of each player (by giving to a & b the same denominations as above) will be

$$\begin{aligned}
&= 1 \times \frac{\overline{n-1}^n}{n^n}\cdot b + \frac{n}{1}\cdot\frac{\overline{n-1}^{n-1}}{n^n}\cdot\frac{A}{n} + \frac{n.\overline{n-1}}{1.2}\cdot\frac{\overline{n-1}^{n-2}}{n^n}\cdot\frac{2A}{n} + \&\text{c.}\cdots \\
&\qquad + 1\cdot\frac{\overline{n-1}^0}{n^n}\times\frac{nA}{n}. \\
&= \left(\tfrac{n-1}{n}\right)^n b \quad + \frac{A}{n^n}\Big(\overline{n-1}^{n-1} + \frac{n-1}{1}\cdot\overline{n-1}^{n-2} + \frac{\overline{n-1}.\overline{n-2}}{1.2}\cdot\overline{n-1}^{n-3} + \&\text{c.} \\
&\qquad + \frac{\overline{n-1}.\overline{n-2}}{1.2}\cdot\overline{n-1}^2 + \frac{n-1}{1}\times\overline{n-1}^1 + 1\Big) \\
&= \left(\tfrac{n-1}{n}\right)^n b \quad + \frac{A}{n^n}(\overline{n-1}+1)^{n-1} = \left(\tfrac{n-1}{n}\right)^n b + \frac{A}{n^n}
\end{aligned}$$

Now the sum A of the lots is equal to the sum of the wagers namely nb less the sum a that the Banker levies first, so that the expectation of each player becomes

$$= \left(\frac{n-1}{n}\right)^n b + b - \frac{a}{n}$$

& the advantage of each player $= \left(\frac{n-1}{n}\right)^n b + b - \frac{a}{n}$ which multiplied by m, will give the disadvantage of the Banker

$$= n \times \left(\frac{n-1}{n}\right)^n \times b - a$$

as by the first method.