OF THE SUMMATION OF RECURRING SERIES

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The Reader may have perceived that the Solution of several Problems relating to Chance depends upon the Summation of Series; I have, as occasion has offered, given the Method of summing them up; but as there are others that may occur, I think it necessary to give a summary View of what is most requisite to be known in this matter; desiring the Reader to excuse me, if I do not give the Demonstrations, which would swell this Tract too much; especially considering that I have already given them in my *Miscellaneous Analytica*.

I call that a recurring Series which is so constituted, that having taken at pleasure any number of its Terms, each following Term shall be related to the same number of preceding Terms, according to a constant law of Relation, such as the following Series

$$\begin{array}{ccccccccccccc} A & & B & & C & & D & & E & & F & & \\ 1 & + & 2x & + & 3xx & + & 10x^3 & + & 34x^4 & + & 97x^5, & \&c. \end{array}$$

in which the Terms being respectively represented by the Capitals A, B, C, D, &c. we shall have

$$D = 3Cx - 2Bxx + 5Ax^3$$
$$E = 3Dx - 2Cxx + 5Bx^3$$
$$F = 3Ex - 2Dxx + 5Cx^3$$
&c.

Now the Quantities $3x - 2xx + 5x^3$, taken together and connected with their proper Signs, is what I call the Index, or the *Scale of Relation*; and sometimes the bare Coefficients $3 - 2 + 5$ are called the Scale of Relation.

PROPOSITION I.

If there be a recurring Series $a + bx + cxx + dx^3 + ex^4$, &c. of which the Scale of Relation be $fx - gxx$; the Sum of that Series continued *in infinitum* will be

$$\frac{\begin{array}{l} a+bx \\ \;\;-fax \end{array}}{1-fx+gxx}$$

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PROPOSITION II.

Supposing that in the Series $a + bx + cxx + dx^3 + ex^4$, &c. the Law of Relation be $fx - gxx + hx^3$; the Sum of that Series continued *in infinitum* will be

$$\frac{\begin{array}{l} a+bx\ \ +cxx \\ \quad -fax-fbxx \\ \qquad\quad +gaxx \end{array}}{1-fx\ +gxx-hx^3}$$

PROPOSITION III.

Supposing that in the Series, $a + bx + cxx$, &c. the Law of Relation be $fx - gxx + hx^3 - kx^4$, the Sum of the Series will be

$$\frac{\begin{array}{l} a+bx\ \ +cxx\ \ +dx^3 \\ \quad -fax-fbxx-fcx^3 \\ \qquad\quad +gaxx+gbx^3 \\ \qquad\qquad\qquad -hax^3 \end{array}}{1-fx\ +gxx\ -hx^3+kx^4}$$

As the Regularity of those Sums is conspicuous, it would be needless to carry them any farther.

Still it is convenient to know that the Relation being given, it will be easy to obtain the Sum by observing this general Rule.

1 $^\circ$, Take as many Terms of the Series as there are parts in the Scale of Relation.

2 $^\circ$, Subtract the Scale of Rotation from Unity, and let the remainder be called the *Differential* Scale.

3 $^\circ$, Multiply those Terms which have been taken in the Series by the Differential Scale, beginning at Unity, and so proceeding orderly, remembering to leave out what would naturally be extended beyond the last of the Terms taken.

Then the Product will be the Numerator of a Fraction expressing the Sum, of which the Denominator will be the Differential Scale.

Thus to form the preceding Theorem,

Multiply $a + bx + cxx + dx^3$
by $1 - fx + gxx - hx^3 \cdots$

and beginning from Unity, we shall have

$$\begin{array}{lllll} a & +bx & +cxx & +dx^3 \\ & -fax & -fbxx & -fcx^3\cdots \\ & & +gaxx & +gbx^3\cdots \\ & & & -hax^3\cdots \end{array}$$

omitting the superfluous Terms, and thus will the Numerator be formed; but the Denominator will be the Differential Scale, *viz.* $1 - fx + gxx - hx^3 + kx^4$.

COROLLARY.

If the first Terms of the Series are not taken at pleasure, but begin from the second Term to follow the Law of Relation, in so much that

$$\begin{array}{lll} b & \text{shall be} & = fa \\ c & & = fb - ga \\ d & & = fc - gb + ha \\ \&c. & & \end{array}$$

then the Fraction expressing the Sum of the Series will have barely the first Term of the Series for its Numerator.

PROPOSITION IV.

If a Series is so constituted, as that the last Differences of the Coefficients of the Terms whereof it is composed be all equal to nothing, the Law of the Relation will be found in the Binomial $(1-x)^n$, n denoting the rank of those last Differences; thus supposing the Series

$$\begin{matrix} A & B & C & D & E & F & G \\ 1 & +4x & +10xx & +20x^3 & +35x^4 & +56x^5 & +84x^5, \quad \&c. \end{matrix}$$

whereof the Coefficients are,

	1	+4	+10	+20	+35	+56	+84
1st Differences		3	+6	+10	+15	+21	+28
2nd Differences			3	+4	+5	+6	+7
3rd Differences				1	+1	+1	+1
4th Differences					0	+0	+0

I say that the Relation of the Terms will be found in the Binomial $(1-x)^4$, which being expanded will be $1-4x+6xx-4x^3+x^4$ and is the Differential Scale, and therefore the Scale properly so called will be $4x-6xx+4x^3-x^4$; thus, in the foregoing Series, the Term

$$G = 4Fx - 6Exx + 4Dx^3 - 1Cx^4.$$

COROLLARY.

The Sums of those infinite Series which begin at Unity, and have their Coefficients the figurate numbers of any order, are always expressible by the Fraction $\frac{1}{(1-x)^p}$, wherein p denotes the rank or order which those figurate numbers obtain; for Instance if we take the Series
$1+1x+1xx+1x^3+1x^4+1x^5+1x^6$, &c. which is a geometric Progression, and whose Coefficients are the numbers of the first order, the Sum will be $\frac{1}{1-x}$, and if we take the Series $1+2x+3xx+4x^3+5x^4+6x^5+7x^6$, &c. whose Coefficients compose the numbers of the second order, the Sum will be $\frac{1}{(1-x)^2}$; and again, if we take the Series $1+3x+6xx+10x^3+15x^4$, &c. whose Coefficients are the numbers of the third order, otherwise called the Triangular numbers, the Sum will be $\frac{1}{(1-x)^3}$.

PROPOSITION V.

The Sum of any finite number of Terms of a recurring Series $a+bx+cxx+dx^3+ex^4$, &c. is always to be obtained.

Thus supposing the Scale of Relation to be $fx-gxx$; n the number of Terms whose Sum is required; and $\alpha x^n + \beta x^{n+1}$ the two Terms which would next follow the last of the given Terms; if the Series was continued; then the Sum will be

$$\frac{\begin{matrix} a+bx - x^n \times \overline{\alpha+\beta x} \\ -fax \qquad\qquad -f\alpha x \end{matrix}}{1-fx+gxx}$$

But if the Scale of Relation be $fx-gxx+hx^3$, n the number of Terms given, and $\alpha x^n+\beta x^{n+1}+\gamma x^{n+2}$, the three Terms that would next follow the last of the given Terms, then the Sum will be

$$\frac{\begin{array}{rr} a+bx+cxx & -x^n\times\overline{\alpha+\beta x+\gamma xx} \\ -fax-fbxx & -f\alpha x-f\beta x \\ +gaxx & +gaxx \end{array}}{1-fx+gxx-hx^3}$$

The continuation of which being obvious, those Theorems need not be carried any farther.

But as there is a particular elegancy for the Sums of a finite number of Terms in those Series whose Coefficients are figurate numbers beginning at Unity, I shall let down the *Canon* for those Sums.

Let n denote the number of Terms whose Sum is to be found, and p the rank or order which those figurate numbers obtain, then the Sum will be

$$\frac{1-x^n}{(1-x)^p}-\frac{nx^n}{(1-x)^{p-1}}-\frac{n.n+1.x^n}{1.2.(1-x)^{p-2}}$$
$$-\frac{n.n+1.n+2.x^n}{1.2.3.(1-x)^{p-3}}-\frac{n.n+1.n+2.n+3}{1.2.3.4(1-x)^{p-4}},\ \&c.$$

which is to be continued till the number of Terms be $= p$.

Thus supposing that the Sum of twelve Terms of the Series, $1+3x+6xx+10x^3+15x^4$, &c. were demanded, that Sum will be

$$\frac{1-x^{12}}{(1-x)^3}-\frac{12x^{12}}{1.2.(1-x)}-\frac{12.13x^{12}}{1.2.3.1-x}.$$

PROPOSITION VI.

In a recurring Series, any Term may be obtained whose place is assigned.

It is very plain, form what we have said, that after having taken so many Terms of the Series as there is in the Scale of Relation, the Series may be protracted till it reach the place assigned; however if that place be very distant from the beginning of the Series, the continuation of those Terms may prove laborious, especially if there be many parts in the Scale.

But there being frequent Cases wherein that inconveniency may be avoided, it will be proper to shew by what Rule this may be known; and then to shew how we are to proceed.

The Rule will be to take the Differential Scale, and to suppose it $= 0$, then if the roots of that supposed Equation be all real, and unequal, the thing may be effected as follows. Let the Series be represented by

$$a+br+crr+dr^3+er^4,\ \&c.$$

and 1° if $fr-grr$ be the Scale of Relation, and consequently $1-fr+grr$ the differential Scale, then having made $1-fr+grr=0$; multiply the Terms of that Scale respectively by xx, x, 1, so as to have $xx-frx+grr=0$, let m and p be the two roots of that Equation, then having made $A=\frac{br-pa}{m-p}$ and $B=\frac{br-ma}{p-m}$, and supposing l to be the interval between the first Term and the place assigned, that Term will be Am^l+Bp^l.

Secondly, If the Scale of Relation be $fr-grr+hr^3$, make $1-fr+grr-hr^3=0$, the Terms of which Equation being multiplied respectively by x^3, xx, x, 1, we shall have the new Equation $x^3-frxx+grrx-hr^3=0$, let m, p, q be the roots of that Equation, then having made $A=\frac{crr-(p+q)\times br+pqa}{(m-p)\times(m-q)}$, $B=\frac{crr-(m+q)\times br+mqa}{(p-m)\times(p-q)}$, $C=$

$\frac{crr-(p+m)\times br+mqa}{(q-m)\times(q-p)}$;

And supposing as before l to be the Interval between the first Term and the Term whose place is assigned, that Term will be $Am^l + Bp^l + Cq^l$.

Thirdly, If the Scale of Relation be $fr - grr + hr^3 - kr^4$ make $1 - fr + grr - hr^3 + kr^4 = 0$, and multiply its Terms respectively by x^4, x^3, xx, x, 1, so as to have the new Equation $x^4 - frx^3 + grrx^2 - hr^3x + kr^4 = 0$, let m, p, q, s, be roots of the Equation, then having made

$$A = \frac{dr^3-(p+q+s)\times crr+(pq+ps+qs)\times br-pqs\times a}{(m-p)\times(m-q)\times(m-s)}$$

$$B = \frac{dr^3-(q+s+m)\times crr+(qs+qm+sm)\times br-qsm\times a}{(p-q)\times(p-s)\times(p-m)}$$

$$C = \frac{dr^3-(s+m+p)\times crr+(sm+sp+mp)\times br-smp\times a}{(q-s)\times(q-m)\times(q-p)}$$

$$D = \frac{dr^3-(m+p+q)\times crr+(mp+mq+pq)\times br-mpq\times a}{(s-m)\times(s-p)\times(s-q)}$$

then, still supposing l to be the Interval between the first Term and the Term whose place is assigned, that Term will be $Am^l + Bp^l + Cq^l + Dx^l$.

Although one may by a narrow inspection perceive the Order of those Theorems, it will not be amiss to express them in words at length.

GENERAL RULE.

Let the Roots m, p, q, s, &c. determined as above, be called respectively, first, second, third, fourth Root, &c. let there be taken as many Terms of the Series beginning from the first, as there are parts in the Scale of Relation: then multiply in an inverted order, 1°, the last of these Terms by Unity; 2°, the last but one by the Sum of the Roots wanting the first; 3°, the last but two, by the Sum of the Products of the Roots taken two and two, excluding that product wherein the first Root is concerned; 4°, the last but three, by the Sum of the Products of the Roots taken three and three, still excluding that Product in which the first Root is concerned, and so on; then all the several parts which are thus generated by Multiplication being connected together by Signs alternately positive and negative, will compose the Numerator of that Fraction to which A is equal; now the Numerator of that Fraction to which B is equal will be formed in the same manner, excluding the second Root instead of the first, and so on.

As for the Denominators, they are formed in this manner: From the first Root subtract severally all the others, and let all the remainders be multiplied together, and the Product will constitute the Denominator of the Fraction to which A is equal; and in the same manner, from the second Root subtracting all the others, let all the remainders be multiplied together, and the Product will constitute the Denominator of the Fraction to which B is equal, and so on for the Rest.

COROLLARY I.

If the Series in which a Term is required to be assigned, be the Quotient of Unity divided by the differential Scale $1 - fr + grr - hr^3 + kr^4$, multiply the Terms of that Scale respectively by x^4, x^3, x^2, x, 1, so as to make the first Index of x equal to the last of r, then make the Product $x^4 - frx^3 + grrxx - hr^3x + kr^4$ to be $= 0$. Let as before m, p, q, s, be the Roots of that Equation, let also z be the number of those Roots, and l the Interval between the first Term, and the Term required, the make

$$A = \frac{m^{z-1}}{(m-p)\times(m-q)\times(m-s)}, \qquad B = \frac{p^{z-1}}{(p-m)\times(p-q)\times(p-s)}$$
$$C = \frac{q^{z-1}}{(q-m)\times(q-p)\times(q-s)}, \qquad D = \frac{s^{z-1}}{(s-m)\times(s-p)\times(s-q)}$$

and the Term required will be $Am^l + Bp^l + Cq^l + Ds^l$; and the Sum of the Terms will be

$$A\times\frac{1-m^{l+1}}{1-m} + B\times\frac{1-p^{l+1}}{1-p} + C\times\frac{1-q^{l+1}}{1-q} + D\times\frac{1-s^{l+1}}{1-s}$$

It is to be observed, that the Interval between the first Term and the Term required is always measured by the number of Terms wanting one, so that having for Instance the Terms, a, b, c, d, e, f, whereof a is the first and f the Term required, the Interval between a and f is 5, and the Number of all the Terms is 6.

COROLLARY 2.

If in the recurring Series $a+br+crr+dr^3+er^4$, &c. whereof the Differential Scale is supposed to be $1-fr+grr-hr^3+kr^4$, we make $x^4-frx^3+grrxx-hr^3x+kr^4=0$, and that the Roots of that Equation be m, p, q, s, and that it so happen that so many Terms of the Series $a+br+crr+dr^3+er^4$, &c. as there are Roots, be every one of them equal to Unity, then any Term of the Series may be obtained thus; let l be the Interval between the first Term and the Term required, make

$$A = \frac{(1-p)\times(1-q)\times(1-s)}{(m-p)\times(m-q)\times(m-s)}, \qquad B = \frac{(1-q)\times(1-s)\times(1-m)}{(p-q)\times(p-s)\times(p-m)}$$
$$C = \frac{(1-s)\times(1-m)\times(1-p)}{(q-s)\times(q-m)\times q-p)}, \qquad D = \frac{(1-m)\times(1-p)\times(1-q)}{(s-m)\times(s-p)\times(s-q)}$$

and the Term required will be $Am^l + Bp^l + Cq^l + Ds^l$.

PROPOSITION VII.

If there be given a recurring Series whose Scale of Relation is $fr - grr$, and out of that Series be composed two other Series, whereof the first shall contain all the Terms of the Series given which is posited in an odd place, and the second shall contain all the Terms that are posited in even place; then the Scale of Relation in each of these two new Series may be obtained as follows:

Take the differential Scale $1 - fr + grr$, out of which compose the Equation $xx - frx + grr = 0$; then making $xx = z$, expunge the Quantity x, whereby the Equation will become $z - fr\sqrt{z} + grr = 0$, or $z + grr = fr\sqrt{z}$; and squaring both parts, to take away the Radicality, we shall have the new Equation $zz + 2grrz + ggr^4 = ffrrz$, or $zz + \begin{matrix} 2grrz \\ -ffrrz \end{matrix} + ggr^4 = 0$; and dividing its Terms respectively by zz, z, 1, we shall have a new differential Scale for each of the two new Series into which the Series given was divided, which will be $1 + \begin{matrix} 2grr \\ -ffrr \end{matrix} + ggr^4$: and this being obtained, it is plain from our first Proposition, that each of the two new Series may be summed up.

But if the Scale of Relation be extended to three Terms, such as the Scale $fr - grr + hr^3$, then the differential Scale for each of the two Series into which

the Series given may be supposed to be divided, will be $1 - ffrr - 2fhr^4 - hhr,$
$+2grr + ggr^4$
whereby it appears that each of the two new Series may be summed up.

If instead of dividing the Series given into two Series, we divide it into three, whereof the first shall be composed of the

1st, 4th, 7th, 10th, &c. Terms; the second of the

2nd, 5th, 8th, 11th, &c. Terms; the third of the

3rd, 6th, 9th, 12th, &c. Terms; and that the Scale of Relation be supposed $fr - grr$; then taking the differential Scale $1 - fr + grr$, and having out of it formed the Equation $xx - frx + grr = 0$, suppose $x^3 = z$; let now x be expunged, and the Equation will be changed into this $zz + 3fgr^3z + g^3r^6 = 0$, of which the Terms being
$-f^3r^3z$
divided respectively by zz, z, 1, we shall have a differential Scale $1 - f^3r^3 + g^3r^6,$
$+3fgr^3$
which will serve for every one of the three Series into which the Series given is divided; and therefore every one of those three Series may be summed up, by help of the two first Terms of each.

If the Scale of Relation be composed of never so many parts, still if the Series given be to be divided into three other Series; from the supposition of x^3 being made $= z$, will be derived a Scale of Relation for the three parts into which the Series given is to be divided.

But if the Series given was to be divided into 4, 5, 6, 7, &c. Series given, suppose accordingly $x^4 = z$, $x^5 = z$, $x^6 = z$, $x^7 = z$, &c. and x being expunged by the common Rules of Algebra, the Scale of Relation will be obtained for every one of the Series into which the Series given is to be divided.

PROPOSITION VIII.

If there be given two Series, each having a particular Scale of Relation, and that the corresponding Terms of both Series be added together, so as to compose a third Series, the differential Scale for this third Series will be obtained as follows.

Let $1 - fr + grr$ be the differential Scale of the first, and $1 - mr + prr$. the differential Scale of the second; let those two Scales be multiplied together, and the Product $1 - \overline{m + f} \times r + \overline{p + g + mf} \times rr - \overline{mg + pf} \times r^3 + pg \times r^4$, will express the differential Scale of the Series resulting from the addition of the other two.

And the same Rule will hold, if one Series be subtracted from the other.

PROPOSITION IX.

If there be given two recurring Series, and that the corresponding Terms of those two Series be multiplied together, the differential Scale of the Series resulting from the Multiplication of the other two may be found as follows.

Suppose $1 - fr + grr$ to be the differential Scale of the first, and $1 - ma + paa$ the differential Scale of the second, so that the first Series shall proceed by the powers of r, and the second by the powers of a; imagine those two differential Scales to be Equations equal to nothing, and both r and a to be indeterminate quantities; make $ar = z$, and now by means of the three Equations, $1 - fr + grr = 0$, $1 - ma + paa = 0$, $ar = z$, let both a and r be expunged, and the Equation resulting from that

Operation will be

$$
\begin{aligned}
&1 - fmz + ffpzz - fgmpz^3 + ggppz^4 = 0\\
&\qquad\quad +mmgzz\\
&\qquad\quad -2gpzz\\
\text{or}\quad &1 - fmar + ffpa^2r^2 - fgmpa^3r^3 + ggppa^4r^4 = 0\\
&\qquad\quad +mmga^2r^2\\
&\qquad\quad -2gpa^2r^2
\end{aligned}
$$

by substituting ar in the room of z; and the Terms of that Equation, without any regard to their being made $= 0$, which was purely a fiction, will express the differential Scale required: and in the same manner may we proceed in all other more compound Cases.

But it is very observable, that if one of the differential Scales be the Binomial $1 - a$ raised to any Power, it will be sufficient to raise the other differential Scale to that Power, only substituting ar for r, or leaving the Powers of r as they are, if a be restrained to Unity; and that Power of the other differential Scale will constitute the differential Scale required.