THE GAME OF RENCONTRE

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Extract from the Preface of 1717.

In the 35th and 36th Problems, I explain a new sort of Algebra, whereby some Questions relating to Combinations are solved by so easy a Process, that their Solution is made in some measure an immediate consequence of the Method of Notation. I will not pretend to say that this new Algebra is absolutely necessary to the Solving of those Questions which I make to depend upon it, since it appears that Mr. Montmort, Author of the Analyse des Jeux de Hazard, and Mr. Nicholas Bernoulli have solved, by another Method, many of the cases therein proposed: But I hope I shall not be thought guilty of too much Confidence, I assure the Reader, that the Method I have followed has a degree of Simplicity, not to say of Generality, which will hardly be attained by any other Steps than by those I have taken.

PROBLEM XXXV.

Any number of Letters a, b, c, d, e, f, &c. all of them different, being taken promiscuously as it happens: to find the Probability that some of them shall be found in their places according to the rank they obtain in the Alphabet; and that others of them shall at the same time be displaced.

SOLUTION

Let the number of all the Letters be $= n$; let the number of those that are to be in their places be $= p$, and the number of those that are to be out of their places $= q$. Suppose for brevity's sake

$$\frac{1}{n} = r, \ \frac{1}{n.n-1} = s, \ \frac{1}{n.n-1.n-2} = t, \ \frac{1}{n.n-1.n-2.n-3} = v, \ \&c.$$

then let all the quantities 1, r, s, t, v, &c. be written down with Signs alternately positive and negative, beginning at 1, if p be $= 0$; at r, if p be $= 1$: at s, if p be $= 2$, &c. Prefix to these Quantities the Coefficients of a Binomial Power, whose index is $= q$; this being done, those Quantities taken all together will express the Probability required. Thus the Probability that in 6 Letters taken promiscuously, two of them, *viz.* a and b shall be in their places, and three of them, *viz.* c, d, e, out of their places, will be

$$\frac{1}{6.5} - \frac{3}{6.5.4} + \frac{3}{6.5.4.3} - \frac{1}{6.5.4.3.2} = \frac{11}{720}$$

And the Probability that a shall be in its place, and b, c, d, e, out of their places, will be

$$\frac{1}{6} - \frac{4}{6.5} + \frac{6}{6.5.4} - \frac{4}{6.5.4.3} + \frac{1}{6.5.4.3.2} = \frac{53}{720}$$

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The Probability that a shall be in its place, and b, c, d, e, f, out of their places, will be

$$= \frac{44}{720} = \frac{11}{180}.$$

The Probability that a, b, c, d, e, f shall all be displaced is

$$1 - \frac{6}{6} + \frac{15}{6.5} - \frac{20}{6.5.4} + \frac{15}{6.5.4.3} - \frac{6}{6.5.4.3.2} + \frac{1}{6.5.4.3.2.1}$$

or

$$1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} = \frac{265}{720} = \frac{53}{144}.$$

Hence it may be concluded, that the Probability that one of them at least shall be in its place, is

$$1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720} = \frac{91}{144},$$

and that the Odds that one of them at least shall be so found, are as 91 to 53.

It must be observed, that the foregoing Expression may serve for any number of Letters, by continuing it to so many Terms as there are Letters: thus if the number of Letters had been seven, the Probability required would have been $\frac{177}{280}$.

DEMONSTRATION.

The number of Chances for the Letter a to be in the first place, contains the number of Chances by which a being in the first place, b may be in the second, or out of it: This is an Axiom of common Sense of the same degree of evidence, as that the whole is equal to all its parts.

From this it follows, that if from the number of Chances that there are for a to be in the first place, there be subtracted the number of Chances that there are for a to be in the first place, and b at the same time in the second, there will remain the number of Chances by which a being in the first place, b may be excluded the second.

For the same reason it follows, that if from the number of Chances for a and b to be respectively in the first and second places, there be subtracted the number of Chances by which a, b, and c may be respectively in the first, second, and third places; there will remain the number of Chances by which a being in the first, and b in the second, c may be excluded the third place: and so of the rest.

Let $+a$ denote the Probability that a shall be in the first place, and let $-a$ denote the Probability of its being out of it. Likewise let the Probabilities that b shall be in the second place, or out of it, be respectively exprest by $+b$ and $-b$.

Let the Probability that a being in the first place, b shall be in the second, be expressed by $a + b$: Likewise let the Probability that a being in the first place, b shall be excluded the second, be expressed by $a - b$.

Universally. Let the Probability there is that as many as are to be in their proper places, shall be so, and that as many others as are at the same time to be out of their proper places shall be so found, be denoted by the particular Probabilities of their being in their proper places, or out of them, written all together: So that, for instance $a + b + c - d - e$, may denote the Probability that a, b, and c shall be in their proper places, and that at the same time both d and e shall be excluded their proper places.

Now to be able to derive proper Conclusions by virtue of this Notation, it is to be observed, that of the Quantities which are here considered, those from which the Subtraction is to be made are indifferently composed of any number of Terms connected by $+$ and $-$; and that the Quantities which are to be subtracted do exceed by one Term those from which the Subtraction is to be made; the rest of the Terms being alike, and their Signs alike; then

nothing more is requisite to have the remainder, than to preserve the Quantities that are alike, with their proper Signs, and to change the Sign of the Quantity exceeding.

It having been demonstrated in what we have said of Permutations and Combinations, that

$$a = \frac{1}{n}; \quad a + b = \frac{1}{n.n-1}; \quad a + b + c = \frac{1}{n.n-1.n-2}, \quad \&c.$$

let $\frac{1}{n}$, $\frac{1}{n.n-1}$, &c. be respectively called r, s, t, v, &c. this being supposed, we may come to the following Conclusions.

$$
\begin{array}{lll}
& b & = r \\
& b + a & = s \\
\text{then 1.} & \overline{b - a} & = r - s \\
& c + b & = s & \text{the same reason that } a + bs. \\
& c + b + a & = t \\
\text{2.} & \overline{c + b - a} & = s - t \\
& c - a & = r - s & \text{by the first Conclusion.} \\
& c - a + b & = +s - t & \text{by the second.} \\
\text{3.} & \overline{c - a - b} & = r - 2s + t \\
& d + c + b & = t \\
& d + c + b + a = v \\
\text{4.} & \overline{d + c + b - a = t - v} \\
& d + c - a & = s - t & \text{by the second Conclusion.} \\
& d + c - a + b = t - v & \text{by the fourth.} \\
\text{5.} & \overline{d + c - a - b = s - 2t + v} \\
& d - b - a & = r - 2s + t & \text{by the third Conclusion.} \\
& d - b - a + c = s - 2t + v & \text{by the fifth.} \\
\text{6.} & \overline{d - b - a - c = r - 3s + 3t - v.}
\end{array}
$$

By the same process, if no letter be particularly assigned to be in its place the Probability that such of them as are assigned may be out of their places, will likewise be found thus.

$$
\begin{array}{lll}
& -a & = 1 - r, & \text{for } +a \text{ and } -a \text{ together make Unity.} \\
& -a + b & = r - s & \text{by the first Conclusion.} \\
\text{7.} & \overline{-a - b} & = 1 - 2r + s \\
& -a - b & = 1 - 2r + s & \text{by the seventh Conclusion.} \\
& -a - b + c = r - 2s + t & \text{by the third.} \\
\text{8.} & \overline{-a - b - c = 1 - 3r + 3s - t.}
\end{array}
$$

Now examining carefully all the foregoing Conclusions, it will be perceived, that when a Question runs barely upon the displacing any given number of Letters, without requiring that any other should be in its place, but leaving it wholly indifferent; then the Vulgar Algebraic Quantities which lie at the right-hand of the Equations, begin constantly with Unity: it will also be perceived, that when one single Letter is assigned to be in its place, then those Quantities begin with r, and that when two Letters are assigned to be in their places, they begin with s, and so on: moreover 'tis obvious, that these Quantities change their Signs alternately, and that the numerical Coefficients, which are prefixed to them are those of a Binomial Power, whose Index is equal to the number of Letters which are to be displaced.

PROBLEM XXXVI.

Any given number of Letters a, b, c, d, e, f, &c. being each repeated a certain number of times, and taken promiscuously as it happens: To find the Probability that of some of

those sorts, some one Letter of each may be found in its place, and at the same time, that
of some other sorts, no one Letter be found in its place.

<div align="center">SOLUTION.</div>

Suppose n be the number of all the Letters, l the number of times that each Letter
is repeated, and consequently $\frac{n}{l}$ the whole number of Sorts: suppose also that p be the
number of Sorts of which some one Letter is to be found in its place, and q the number of
Sorts of which no one Letter is to be found in its place. Let now the prescriptions given in
the preceding Problem be followed in all respects, saving that r must here be made

$$= \frac{l}{n}, \quad s = \frac{ll}{n.n-1}, \quad t = \frac{l^3}{n.n-1.n-2}, \quad \text{\&c.}$$

and the Solution of any particular case of the Problem will be obtained.

Thus if it were required to find the Probability that no Letter of any sort shall be in its
place, the Probability thereof would be expressed by the Series

$$1 - qr + \frac{q.q-1}{1.2}s - \frac{q.q-1.q-2}{1.2.3}t + \frac{q.q-1.q-2.q-3}{1.2.3.4}v, \text{ \&c.}$$

of which the number of Terms is equal to $q+1$.

But in this particular case q would be equal to $\frac{n}{l}$, and therefore, the foregoing Series
might be changed into this, *viz.*

$$\frac{1}{2} \times \frac{n-l}{n-1} - \frac{1}{6} \times \frac{n-l.n-2l}{n-1.n-2} + \frac{1}{24} \times \frac{n-l.n-2l.n-3l}{n-1.n-2.n-3}, \text{ \&c.}$$

of which the number of Terms is equal to $\frac{n-l}{l}$.

<div align="center">COROLLARY 1.</div>

From hence it follows, that the Probability of one or more Letters, indeterminately
taken, being in their places, will be expressed as follows:

$$1 - \frac{1}{2} \times \frac{n-l}{n-1} + \frac{1}{6} \times \frac{n-l.n-2l}{n-1.n-2} - \frac{1}{24} \times \frac{n-l.n-2l.n-3l}{n-1.n-2.n-3}, \text{ \&c}$$

<div align="center">COROLLARY 2.</div>

The Probability of two or more Letters indeterminately taken, being in their places, will
be

$$\frac{1}{2} \times \frac{n-l}{n-1} - \frac{2}{1.3} \times \frac{n-2l}{n-2}A - \frac{3}{2.4} \times \frac{n-3l}{n-3}B$$
$$- \frac{4}{3.5} \times \frac{n-4l}{n-4}C + \frac{5}{4.6} \times \frac{n-5l}{n-5}D, \text{ \&c.}$$

wherein it is necessary to observe, that the Capitals A, B, C, D, &c. denote the preceding
Terms.

Altho' the formation of this last Series flows naturally from what we have already estab-
lished, yet nothing may be wanting to clear up this matter, it is to be observed, that if one
Species is to have some one of its Letters in its proper place, and the rest of the Species
to be excluded, then the Series whereby the Problem is determined being to begin at r,
according to the Precepts given in the preceding Problem, becomes

$$r - qs + \frac{q.q-1}{1.2}t - \frac{q.q-1.q-2}{1.2.3}v, \text{ \&c.}$$

but then the number of Species being $\frac{n}{l}$, and all but one being to be excluded, it follows
that q in this case is $= \frac{n}{l} - 1 = \frac{n-l}{l}$ wherefore the preceding Series would become, after

the proper Substitutions,

$$\frac{l}{n} - \frac{n-l.l}{n.n-1} + \frac{1}{2} \times \frac{n-l.n-2l.l}{n.n-1.n-2} - \frac{1}{6} \times \frac{n-l.n-2l.n-3l.l}{n.n-1.n-2.n-3}, \ \&c.$$

And this is the Probability that some one of the Letters of the Species particularly given, may obtain its place, and the rest of the Species be excluded; but the number of Species being $\frac{n}{l}$, it follows that this Series ought to be multiplied by $\frac{n}{l}$, which will make it

$$1 - \frac{n-l}{n-1} + \frac{1}{2} \times \frac{n-l.n-2l}{n-1.n-2} - \frac{1}{6} \times \frac{n-l.n-2l.n-3l}{n-1.n-2.n-3}, \ \&c.$$

And this is the Probability that some one Species indeterminately taken, and not more than one, may have some one of its Letters in its proper place.

Now if from the Probability of one or more being in their places, be subtracted the Probability of one and no more being in its place, there will remain the Probability of two or more indeterminately taken being in their places, which consequently will be the difference between the following Series, *viz.*

$$1 - \frac{1}{2} \times \frac{n-l}{n-1} + \frac{1}{6} \times \frac{n-l.n-2l}{n-1.n-2} - \frac{1}{24} \times \frac{n-l.n-2l.n-3l}{n-1.n-2.n-3}, \ \&c.$$

and

$$1 - \frac{n-l}{n-1} + \frac{1}{2} \times \frac{n-l.n-2l}{n-1.n-2} - \frac{1}{6} \times \frac{n-l.n-2l.n-3l}{n-1.n-2.n-3}, \ \&c.$$

which difference will be

$$\frac{1}{2} \times \frac{n-l}{n-1} - \frac{1}{3} \times \frac{n-l.n-2l}{n-1.n-2} + \frac{1}{8} \times \frac{n-l.n-2l.n-3l}{n-1.n-2.n-3}, \ \&c.$$

or

$$\frac{1}{2} \times \frac{n-l}{n-1} - \frac{2}{1.3} \times \frac{n-2l}{n-2}A + \frac{3}{2.4} \times \frac{n-3l}{n-3}B - \frac{4}{3.5} \times \frac{n-4l}{n-4}C, \ \&c.$$

as we had expressed it before: and from the same way of reasoning, the other following Corollaries may be deduced.

<div align="center">COROLLARY 3.</div>

The Probability that three or more Letters indeterminately taken may be in their places, will be expressed by the Series

$$\frac{1}{6} \times \frac{n-l.n-2l}{n-1.n-2} - \frac{3}{1.4} \times \frac{n-3l}{n-3}A + \frac{4}{2.5} \times \frac{n-4l}{n-4}B$$
$$- \frac{5}{3.6} \times \frac{n-5l}{n-5}C + \frac{6}{4.7} \times \frac{n-6l}{n-6}D, \ \&c.$$

<div align="center">COROLLARY 4.</div>

The Probability that four or more Letters indeterminately taken may be in their places will be thus expressed

$$\frac{1}{24} \times \frac{n-l.n-2l.n-3l}{n-1.n-2.n-3} - \frac{4}{1.5} \times \frac{n-4l}{n-4}A$$
$$+ \frac{5}{2.6} \times \frac{n-5l}{n-5}B - \frac{6}{3.7} \times \frac{n-6l}{n-6}C, \ \&c.$$

The Law of the continuation of these Series being manifest, it will always be easy to assign one that shall fit any case proposed.

From what we have said it follows, that in a common Pack of 52 Cards, the Probability that one of the four Aces may be in the first place, or one of the four Deuces in the second, or one of the four Trays in the third; or that some of any other sort may be in its place

(making 13 different places in all) will be expressed by the Series exhibited in the first Corollary.

It follows likewise, that if there be two Packs of Cards, and that the order of the Cards in one of the Packs be the Rule whereby to estimate the rank which the Cards of the same suit and Name are to obtain in the other; the Probability that one Card or more in one of the Packs may be found in the same position as the like Card in the other Pack, will be expressed by the Series belonging to the first Corollary, making $n = 52$, and $l = 1$. Which Series will in this case be

$$1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \frac{1}{120} - \frac{1}{720}, \text{ &c.}$$

52 Terms are to be taken.

If the Terms of the foregoing Series are joined by Couples, the Series will become

$$\frac{1}{2} + \frac{1}{2.4} + \frac{1}{2.3.4.6} + \frac{1}{2.3.4.5.6.8} + \frac{1}{2.3.4.5.6.7.8.10} \text{ &c.}$$

of which 26 Terms ought be taken.

But by reason of the great Convergency of the said Series, a few of its Terms will give a sufficient approximation, in all cases; as appears by the following Operation

$$\frac{1}{2} = 0.500000$$

$$\frac{1}{2.4} = 0.125000$$

$$\frac{1}{2.3.4.6} = 0.006944+$$

$$\frac{1}{2.3.4.5.6.8} = 0.000174+$$

$$\frac{1}{2.3.4.5.6.7.8.10} = 0.000002+$$

$$\text{Sum} = 0.632120+$$

Wherefore the Probability that one or more like Cards in two different Packs may obtain the same position, is very nearly 0.632, and the Odds that this will happen once at least as 632 to 368, or 12 to 7 very near.

But the Odds that two or more like Cards in two different Packs will not obtain the same position are very nearly as 736 to 264, or 14 to 5.

REMARK.

It is known that $1 + y + \frac{1}{2}yy + \frac{1}{6}y^3 + \frac{1}{24}y^4$ &c. is the number whose hyperbolic Logarithm is y, and therefore $1 - y + \frac{1}{2}yy - \frac{1}{6}y^3 + \frac{1}{24}y^4$ &c. is the Number whose hyperbolic Logarithm is $-y$. Let N be $= y - \frac{1}{2}yy + \frac{1}{6}y^3 - \frac{1}{24}y^4$ &c. then $1 - N$ is the Number whose hyperbolic Logarithm is $-y$. Let now y be $= 1$, therefore $1 - N$ is the number whose hyperbolic Logarithm is -1; but the number whose hyperbolic Logarithm is -1, is the reciprocal of that whose hyperbolic Logarithm is 1, or whose *Briggian* Logarithm is 0.4342944. Therefore 9.5657056 is the *Briggian* Logarithm answering to the hyperbolic Logarithm -1, but the number answering to it is 0.36788. Therefore $1 - N = 0.36788$; and $N = 1 - 0.36788 = 0.63212$; and therefore the Series

$$y - \frac{1}{2}yy + \frac{1}{6}y^3 - \frac{1}{24}y^4 \text{ &c. } \textit{to infinity,}$$

when $y = 1$, that is

$$1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} \text{ \&c. } = 0.63212.$$

COROLLARY 5.

If A and B each holding a Pack of Cards, pull them out at the same time one after another, on condition that every time two like Cards are pulled out, A shall give B a Guinea; and it were required to find what consideration B ought to give A to play on those Terms: the Answer will be one Guinea, let the number of Cards be what it will.

Altho' this be a Corollary from the preceding Solutions, yet it may more easily be made out thus; one of the Packs being the Rule whereby to estimate the order of the Cards in the second, the Probability that the two first Cards are alike is $\frac{1}{52}$, the Probability that the two second are alike is also $\frac{1}{52}$, and therefore there being 52 such alike combinations, it follows that the Value of the whole is $\frac{52}{52} = 1$.

COROLLARY 6.

If the number of Packs be given, the Probability that any given number of Circumstances may happen in any number of Packs will easily be found by our Method: thus if the number of Packs be k, the Probability that one Card or more of the same Suit and Name in every one of the Packs may be in the same position, will be expressed as follows,

$$\frac{1}{n^{k-2}} - \frac{1}{(2.n.n-1)^{k-2}} + \frac{1}{(6.n.n-1.n-2)^{k-2}} - \frac{1}{(24.n.n-1.n-2.n-3)^{k-2}}, \text{ \&c.}$$