DICING PROBLEMS

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From
De Mensura Sortis, 1711

PROBLEM 18.

A contends with B that he shall throw, in a given number of trials with a die having a given number of faces, certain given faces; A's expectation is sought.

Solution. Let $p+1$ be the number of faces on the die, n the given number of trials, f the number of faces which he must throw.

The number of chances by which A may throw an ace once or more in n trials is $(p+1)^n - p^n$as is clear from what has been previously demonstrated.

Let the deuce be expunged from the number of faces so that the number of faces reduced to p, then the number of chances by which A may throw an ace once or more times in n trials will be $p^n - (p-1)^n$.

Therefore with the deuce restored, the number of chances by which A may throw an ace and a deuce is the difference of these chances, namely $(p+1)^n - 2p^n + (p-1)^n$.

Now let the three be expunged; then the number of chances by which A may throw an ace and a deuce will be $p^n - 2 \times (p-1)^n + (p-2)^n$.

Therefore, with the three restored, the number of chances by which A may throw an ace, a deuce and a three is $(p+1)^n - 3 \times p^n + 3 \times (p-1)^n - (p-2)^n$. And thus successively for the rest.

Therefore, let all the powers be written in order (the signs changing alternately), $(p+1)^n - p^n + (p-1)^n - (p-2)^n + (p-3)^n$, etc. And let the coefficients of a binomial raised to the power f be prefixed to these, and the sum of the terms will be the numerator of A's expectation, the denominator of which will be $(p+1)^n$.

EXAMPLE 1. Let 6 be the number of faces on the die, and 2 the number of given faces which ought to be thrown in 8 trials, then A's expectation will be $(6^8 - 2 \times 5^8 + 4^8)/6^8$.

EXAMPLE 2. Let 6 be the number of faces on the die and 6 the number of faces which ought to be thrown in 12 trials; then A's expectation will be

$$(6^{12} - 6 \times 5^{12} + 15 \times 4^{12} - 20 \times 3^{12} + 15 \times 2^{12} - 6 \times 1^{12})/6^{12}.$$

EXAMPLE 3. A contends with B that he shall throw two given faces in 43 trials with a die having 36 faces, or that he shall throw two aces at the same time on two common dice and also deuces at the same time, then A's expectation will be

$$(36^{43} - 2 \times 35^{43} + 34^{43})/36^{43}.$$

Note that computation of the parts of which these expectations are composed by addition and subtraction will be easy with the help of a Table of Logarithms.

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The Doctrine of Chances, 3rd. Ed., 1756, pp. 123–127

PROBLEM XXXIX.

To find the Expectation of A, when with a Die of any given number of Faces, he undertakes to fling any number of them in any given number of Casts.

SOLUTION.

Let $p+1$ be the number of all the Faces in the Die, n the number of Casts, f the number of Faces which he undertakes to fling.

The number of Chances for the *Ace* to come up once or more in any number of Casts, n, is $(p+1)^n - p^n$: as has been proved in the Introduction.

Let the *Deux*, by thought, be expunged out of the Die, and thereby the number of its Faces reduced to p, then the number of Chances for the *Ace* to come up will at the same time be reduced to $p^n - (p-1)^n$. Let now the *Deux* be restored, and the number of Chances for the *Ace* to come up without the *Deux*, will be the same as if the *Deux* were expunged: But if from the number Chances for the *Ace* to come up with or without the *Deux*, viz. from $(p+1)^n - p^n$ be subtracted the number of Chances for the *Ace* to come up without the *Deux*, viz. $p^n - (p-1)^n$, there will remain the number of Chances for the *Ace* and the *Deux* to come up once or more in the given number of Casts, which number of Chances consequently will be $(p+1)^n - 2p^n + (p-1)^n$.

By the same way of arguing it will be proved, that the number of Chances, for the *Ace* and *Deux* to come up without the *Tray*, will be $p^n - 2\times(p-1)^n + (p-2)^n$, and consequently that the number of Chances for the *Ace*, the *Deux*, and *Tray* to come up once or more, will be the difference between $(p+1)^n - 2p^n + (p-1)^n$, and $p^n - 2\times(p-1)^n + (p-2)^n$, which therefore will be $(p+1)^n - 3\times p^n + 3\times(p-1)^n - (p-2)^n$.

Again, it may be proved that the number of Chances for the *Ace*, the *Deux*, the *Tray*, and the *Quatre* to come up is $(p+1)^n - 4\times p^n + 6\times(p-1)^n - 4\times(p-2)^n + (p-3)^n$; the continuation of which process is manifest.

Wherefore if all the Powers $(p+1)^n$, p^n, $(p-1)^n$, $(p-2)^n$, $(p-3)^n$, &c. with Signs alternately positive and negative be written in order, and to those Powers there be prefixed the respective Coefficients of a Binomial raised to the Power f, expressing the number of Faces required to come up; the Sum of all those Terms will be the Numerator of the Expectation of A, of which the Denominator will be s $(p+1)^n$.

EXAMPLE 1.

The Six be the number of Faces in the Die, and let A undertake in eight Casts to fling both an *Ace* and a *Deux*, without any regard to order: then his Expectation will be $\frac{6^8 - 2\times 5^8 + 4^8}{6^8} = \frac{964502}{1680216} = \frac{4}{7}$ nearly.

EXAMPLE 2.

Let A undertake with a common Die to fling all the Faces in 12 Casts, then his Expectation will be found to be

$$\frac{6^{12} - 6\times 5^{12} + 15\times 4^{12} - 20\times 3^{12} + 15\times 2^{12} - 6\times 1^{12} + 1\times 0^{12}}{6^{12}} = \frac{10}{23}$$

nearly.

EXAMPLE 3.

If A with a Die of 36 Faces undertake to fling two given Faces in 43 Casts; or which is the same thing, if with two common Dice he undertake in 43 Casts to fling two Aces at one time, and two Sixes at another time; his Expectation will be

$$\frac{36^{43} - 2 \times 35^{43} + 34^{43}}{30^{43}} = \frac{49}{100}$$

nearly.

N.B. The parts which compose these Expectations are easily obtained by the help of a Table of Logarithms.

PROBLEM XL.

To find in how many Trials it will be probable that A with a Die of any given number of Faces shall throw any proposed number of them.

Let $p+1$ be the number of Faces in the Die, and f the number of Faces which are to be thrown: Divide the Logarithm of $\frac{1}{1-\sqrt[f]{\frac{1}{2}}}$ by the Logarithm of $\frac{p+1}{p}$, and the Quotient will express the number of Trials requisite to make it as probable that the proposed Faces may be thrown as not.

DEMONSTRATION.

Suppose Six to be the number of Faces that are to be thrown, and n the number of Trials, then by what has been demonstated in the preceding Problem the Expectation of A will be

$$\frac{(p+1)^n - 6 \times p^n + 15 \times (p-1)^n + 20 \times (p-2)^n - 15 \times (p-3)^n + 6 \times (p-4)^n + (p-5)^n}{(p+1)^n}$$

Let it be supposed that the Terms, $p+1$, p, $p-1$, $p-2$, &c. are in geometric Progression, (which Supposition will very little err from the truth, especially if the proportion p to 1, be not very small.) Let now r be written instead of $\frac{p+1}{p}$, and then the Expectation of A will be changed into $1 - \frac{6}{r^n} + \frac{15}{r^{2n}} - \frac{20}{r^{3n}} + \frac{15}{r^{4n}} - \frac{6}{r^{5n}} + \frac{1}{r^{6n}}$, or $\left(1 - \frac{1}{r^n}\right)^6$. But this Expectation of A ought to be made equal to $\frac{1}{2}$, since by Supposition he has an equal Chance to win or lose, hence will arise the Equation $\left(1 - \frac{1}{r^n}\right)^6 = \frac{1}{2}$ or $r^n = \frac{1}{1-\sqrt[6]{\frac{1}{2}}}$, from which it may be concluded that $n \log . r$, or $n \times \log . \frac{p+1}{p} = \log . \frac{1}{1-\sqrt[6]{\frac{1}{2}}}$, and consequently that n is equal to the Logarithm of $\frac{1}{1-\sqrt[6]{\frac{1}{2}}}$, divided by the Logarithm of $\frac{p+1}{p}$. And the same demonstration will hold in any other Case.

EXAMPLE 1.

To find in how many Trials A may with equal Chance undertake to throw all the Faces of a common Die.

The Logarithm of $\frac{1}{1-\sqrt[6]{\frac{1}{2}}} = 0.9621753$; the Logarithm of $\frac{p+1}{p}$ or $\frac{6}{5} = 0.0791812$: wherefore $n = \frac{0.9621753}{0.0791812} = 12+$. From hence it may be concluded, that in 12 Casts A has the worst of the Lay, and in 13 the best of it.

EXAMPLE 2.

To find in how many Trials A may with equal Chance with a Die of thirty-six Faces undertake to throw six determinate Faces; or, in how many Trials he may with a pair of common Dice undertake to throw all the Doublets.

The Logarithm of $\frac{1}{1-\sqrt[6]{\frac{1}{2}}}$ being 0.9621753, and the Logarithm of $\frac{p+1}{p}$ or $\frac{36}{35}$ being 0.0122345; if follows that the number of Casts requisite to that effect is $\frac{0.9621753}{0.0122345}$, or 79 nearly.

But if it were the Law of the Play, that the Doublets must be thrown in a given order, and that any Doublet happening to be thrown out of its Turn should go for nothing; then the throwing of the six Doublets would be like the throwing of the two Aces six times: to produce which, the number of Casts requisite would be found by multiplying 35 by 5.668, as appears from the Table annexed to our v$^{\text{th}}$ Problem; and consequently would be about 198.

N.B. The Fraction $\frac{1}{1-\sqrt[f]{\frac{1}{2}}}$, may be reduced to another form viz. $\frac{\sqrt[f]{2}}{\sqrt[f]{2}-1}$; which will facilitate the taking of its Logarithm.