III. SOLUTIO GENERALIS ALTERA PRAECEDENTIS PROBLEMATIS, OPE COMBINATIONUM & SERIERUM INFINITARUM,

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If two gamesters B & C compete at the same time, B to be designated the victor, C the defeated, let BC be written; and in turn C to be designated the victor, B the defeated; let CB be written: & thus of the remaining.[1]

Let be put 1 ° B to defeat A, and the contest to be concluded in three games

$$\left.\begin{array}{c} BA \\ \hline BC \\ BD \end{array}\right\} \text{Thus it is clear } B \text{ to escape the victor necessarily.}$$

Let be put 2 ° B to defeat A, and the contest to be concluded in four games.

$$\left.\begin{array}{c} BA \\ \hline CB \\ CD \\ CA \end{array}\right\} \text{Thus it is clear } C \text{ to escape the victor necessarily.}$$

Let be put 3 ° B to defeat A, and the contest to be concluded in five games.

$$\left.\begin{array}{cc} BA & BA \\ \hline CB^* & BC \\ DC & DB \\ DA & DA \\ DB & DC \end{array}\right\} \begin{array}{l}\text{Thus it is clear } D \text{ to escape the victor necessarily,} \\ \text{and it in two ways.}^2\end{array}$$

Let be put 4 ° B the first in turn to defeat A, and the contest to be concluded in six games.

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[1] See also pages 139–150 of the *Doctrine of Chances*, 3rd Edition.

BA	BA	BA
CB	CB^*	BC
DC^*	CD^*	DB
AD	AC	AC
AB	AB	AD
AC	AD	AB

Thus it is clear A to escape the victor necessarily, and it in three ways.

Let be put 5° the contest to be concluded in seven games, and let be put always B first to defeat in turn the one A.

BA	BA	BA	BA	BA
CB	CB	CB^*	BC	BC
DC	DC^*	CD	DB	DB
AD^*	DA	AC	AD^*	DA
BA	BD	BA	CA	CD
BC	BC^*	BD	CB	CB
BD	BA	BC	CD	CA

Thus it is clear B or C to escape the victor necessarily, B in three ways, and C in two.

Let be put 6° the contest to be concluded in eight games,

BA	BA	BA	BA	BA	BA	BA	BA
CB	CB	CB	CB	CB^*	BC	BC	BC
DC	DC	DC^*	CD	CD	DB	DB	DB
AD	AD^*	DA	AC	AC	AD	AD^*	DA
BA^*	AB	BD	BA^*	AB	CA^*	AC	CD
CB	CA	CB	DB	DA	BC	BA	BC
CD	CD	CA	DC	DB	BD	BD	BA
CA	CB	CD	DA	DC	BA	BC	BD

Thus it is clear C to escape the victor in three, D in two, B in three ways, &c. Now the letters may be written in order in which the victors are designated.

3	$1B$
4	$1C$
5	$2D$
6	$3A$
7	$3B + 2C$
8	$3C + 2D + 3B$
9	$3D + 2A + 3C + 3D + 2A$
10	$3A + 2B + 3D + 3A + 2B + 3A + 2C + 2D$
&c.	

With the formation of those examined, it will be clear 1° the letter B to be found in any row always as many times, as A is found in the last & penultimate row: 2° C to be found in any row as many times as B in the last row & D in the penultimate row are found: 3° D to be found in any row as many times as C in the last & B in the penultimate: 4° A to be found in any row always as many times as D in the last row & C in the penultimate are found.

But the number of variations corresponding to whatever given number of games, is the double of the number of all variations corresponding to the given number of games diminished by unity: and precisely the Probability which Gamester B has that he may defeat in a given number of games, is one-half the probability which A had that he defeated in the given number of games less one; and furthermore one-fourth the probability which the same A had, that he defeated in the given number of games less two: & thus with the others.

The probability C has, that he may defeat in a given number of games, is one-half the probability which B had, that he defeated in the given number of games less one; and furthermore one-fourth the probability which D had, that he defeated in the given number of games less two.

The probability which D has that he may defeat in a given number of games, is one-half the probability which C had, that he defeated in the given number of games less one; and furthermore one-fourth the probability which B had, that he defeated in the given number of games less two.

The probability which A has that he may defeat in a given number of games, is one-half the probability which D had, that he defeated in the given number of games less one; and furthermore one-fourth the probability which C had that he defeated in the given number of games less two.

Now out of the observations it is easy to compose the Table of Probabilities, which B, C, D, A have that they escape the victorious in a given number of games, and furthermore of those lots or expectations.

Table of Probabilites, &c.

		B	C	D	A
i	3	$\frac{1}{4}\times\overline{4+3p}$			
ii	4		$\frac{1}{8}\times\overline{4+4p}$		
iii	5			$\frac{2}{16}\times\overline{4+5p}$	
iiii	6				$\frac{3}{32}\times\overline{4+6p}$
v	7	$\frac{3}{64}\times\overline{4+7p}$	$\frac{2}{64}\times\overline{4+7p}$		
vi	8	$\frac{3}{128}\times\overline{4+8p}$	$\frac{3}{128}\times\overline{4+8p}$	$\frac{2}{128}\times\overline{4+8p}$	
vii	9		$\frac{3}{256}\times\overline{4+9p}$	$\frac{6}{256}\times\overline{4+9p}$	$\frac{4}{256}\times\overline{4+9p}$
viii	10	$\frac{4}{512}\times\overline{4+10p}$	$\frac{2}{512}\times\overline{4+10p}$	$\frac{6}{512}\times\overline{4+10p}$	$\frac{9}{512}\times\overline{4+10p}$
ix	11	$\frac{13}{1024}\times\overline{4+11p}$	$\frac{10}{1024}\times\overline{4+11p}$	$\frac{2}{1024}\times\overline{4+11p}$	$\frac{9}{1024}\times\overline{4+11p}$
x	12	$\frac{18}{2048}\times\overline{4+12p}$	$\frac{19}{2048}\times\overline{4+12p}$	$\frac{14}{2048}\times\overline{4+12p}$	$\frac{4}{2048}\times\overline{4+12p}$
		&c.	*&c.*	*&c.*	*&c.*

Now indeed these series are converging, and they are able to be summed precisely by common Arithmetic; & either the precise sums if they can be, or at least approximate, if it should not be permitted, will be obtained to use many terms.

To discover the sums of the probabilities proceeding all the way to infinity, which the Gamesters have that they may escape victorious.

Let there be all the Probabilities of the one B to infinity, certainly

$$B' + B'' + B''' + B'''' + B^v + B^{vi} \text{ \&c.} = y$$

The probabilities of the one C

$$C' + C'' + C''' + C'''' + C^v + C^{vi} \text{ \&c.} = z$$

The probabilities of the one D

$$D' + D'' + D''' + D'''' + D^v + D^{vi} \text{ \&c.} = v$$

The probabilities of the one A

$$A' + A'' + A''' + A'''' + A^v + A^{vi} \text{ \&c.} = x$$

But they should be written in a descending perpendicular Scale, according to this way.

$$\begin{array}{lll}
 & B' & = B' \\
 & B'' & = B'' \\
 & B''' & = \frac{1}{2}A'' + \frac{1}{4}A' \\
 & B'''' & = \frac{1}{2}A''' + \frac{1}{4}A'' \\
 & B^v & = \frac{1}{2}A'''' + \frac{1}{4}A''' \quad \text{Hence } y = \frac{1}{4} + \frac{3}{4}x. \\
 & B^{vi} & = \frac{1}{2}A^v + \frac{1}{4}A'''' \\
\text{Therefore} & y & = \frac{1}{4} + \frac{1}{2}x + \frac{1}{4}x.
\end{array}$$

Demonstration.

As a matter of fact the first perpendicular column $= y$, by Hypothesis. It is true $A' + A'' + A''' + A'''' + A^v + A^{vi}$ &c.$= x$, by hypothesis; Therefore

$$\frac{1}{2}A' + \frac{1}{2}A'' + \frac{1}{2}A''' + \frac{1}{2}A'''' + \frac{1}{2}A^v, \&c. = \frac{1}{2}x.$$

Hence

$$\frac{1}{2}A'' + \frac{1}{2}A''' + \frac{1}{2}A'''' + \frac{1}{2}A^v \text{ \&c.} = \frac{1}{2}x - \frac{1}{2}A'.$$

And

$$B' + B'' + \frac{1}{2}A'' + \frac{1}{2}A''' + \frac{1}{2}A'''' + \frac{1}{2}A^v \text{ \&c.} = \frac{1}{2}x - \frac{1}{2}A' + B' + B''.$$

But $\frac{1}{2}A' = 0,\ B'' = 0$ & $B' = \frac{1}{4}$, as it is clear from the Table.
Therefore the second perpendicular column $= \frac{1}{4} + \frac{1}{2}x$.
But the third perpendicular column $= \frac{1}{4}x$.
Therefore there will be $y = \frac{1}{4} + \frac{3}{4}x$.

In a similar manner there may be written

$$\begin{aligned} C' &= C' \\ C'' &= C'' \\ C''' &= \tfrac{1}{2}B'' + \tfrac{1}{4}D' \\ C'''' &= \tfrac{1}{2}B''' + \tfrac{1}{4}D'' \qquad \text{this is } z = \tfrac{1}{2}y + \tfrac{1}{4}v. \\ C^{v} &= \tfrac{1}{2}B'''' + \tfrac{1}{4}D''' \\ C^{vi} &= \tfrac{1}{2}B^{v} + \tfrac{1}{4}D'''' \\ \&\text{c.} &= \\ \text{Therefore} \quad z &= \tfrac{1}{8} + \tfrac{1}{2}y - \tfrac{1}{8} + \tfrac{1}{4}v. \end{aligned}$$

Furthermore there may be written

$$\begin{aligned} D' &= D' \\ D'' &= D'' \\ D''' &= \tfrac{1}{2}C'' + \tfrac{1}{4}B' \\ D'''' &= \tfrac{1}{2}C''' + \tfrac{1}{4}B'' \qquad \text{\& by a like argument it is clear} \\ D^{v} &= \tfrac{1}{2}C'''' + \tfrac{1}{4}B''' \qquad v = \tfrac{1}{2}z + \tfrac{1}{4}y. \\ D^{vi} &= \tfrac{1}{2}C^{v} + \tfrac{1}{4}B'''' \\ \&\text{c.} &= \end{aligned}$$

Finally there may be written

$$\begin{aligned} A' &= A' \\ A'' &= A'' \\ A''' &= \tfrac{1}{2}D'' + \tfrac{1}{4}C' \\ A'''' &= \tfrac{1}{2}D''' + \tfrac{1}{4}C'' \\ A^{v} &= \tfrac{1}{2}D'''' + \tfrac{1}{4}C''' \qquad \text{Whence } x = \tfrac{1}{2}v + \tfrac{1}{4}z \text{ will be concluded.} \\ A^{vi} &= \tfrac{1}{2}D^{v} + \tfrac{1}{4}C'''' \\ \&\text{c.} &= \end{aligned}$$

But with these four equations resolved, it will be discovered

$$\begin{aligned} B' + B'' + B''' + B'''' \ \&\text{c.} &= y = \tfrac{56}{149} \\ C' + C'' + C''' + C'''' \ \&\text{c.} &= z = \tfrac{36}{149} \\ D' + D'' + D''' + D'''' \ \&\text{c.} &= v = \tfrac{32}{149} \\ A' + A'' + A''' + A'''' \ \&\text{c.} &= x = \tfrac{25}{149} \end{aligned}$$

With those values discovered, now let be put $\frac{56}{149} = b$, $\frac{36}{149} = c$, $\frac{32}{149} = d$, $\frac{25}{149} = a$.

Again let there be.

$$\begin{aligned} 3B'p + 4B''p + 5B'''p + 6B''''p \ \&\text{c.} &= py \\ 3C'p + 4C''p + 5C'''p + 6C''''p \ \&\text{c.} &= pz \\ 3D'p + 4D''p + 5D'''p + 6D''''p \ \&\text{c.} &= pv \\ 3A'p + 4A''p + 5A'''p + 6A''''p \ \&\text{c.} &= px \end{aligned}$$

$$\begin{aligned} 3B' &= 3B' \\ 4B'' &= 4B'' \\ 5B''' &= \tfrac{5}{2}A'' + \tfrac{5}{4}A' \\ 6B'''' &= \tfrac{6}{2}A''' + \tfrac{6}{4}A'' \\ 7B^{v} &= \tfrac{7}{2}A'''' + \tfrac{7}{4}A''' \\ 8B^{vi} &= \tfrac{8}{2}A^{v} + \tfrac{8}{4}A'''' \\ \text{Therefore}\quad y &= \tfrac{3}{4} + \tfrac{3}{4}x + a. \end{aligned}$$

As a matter of fact the first perpendicular Column $= y$, by Hypothesis:

$3B' + 4B'' = \frac{3}{4}$: For there is $B' = \frac{1}{4}$, & $B'' = 0$.
$3A' + 4A'' + 5A'''$ &c. $= x$ by Hypothesis.
$A' + A'' + A'''$ &c. $= a$, as was discovered.

There is therefore $4A' + 5A'' + 6A''' + 7A''''$ &c. $= x + a$
And $\frac{4}{2}A' + \frac{5}{2}A'' + \frac{6}{2}A''' + \frac{7}{2}A''''$ &c. $= \frac{1}{2}x + \frac{1}{2}a$.
But $A' = 0$.
Therefore the second perpendicular column $= \frac{3}{4} + \frac{1}{2}x + \frac{1}{2}a$.
$3A' + 4A'' + 5A''' + 6A''''$ &c. $= x$
$2A' + 2A'' + 2A''' + 2A''''$ &c. $= 2a$
There is therefore $5A' + 6A'' + 7A''' + 8A''''$ &c. $= x + 2a$
And $\frac{5}{2}A' + \frac{6}{2}A'' + \frac{7}{2}A''' + \frac{8}{2}A''''$ &c. $= \frac{1}{4}x + \frac{1}{2}a$.
Therefore the third perpendicular column is $= \frac{1}{4}x + \frac{1}{2}a$.
Therefore there will be $y = \frac{3}{4} + \frac{1}{2}x + \frac{1}{2}a + \frac{1}{4}x + \frac{1}{2}a$
or $y = \frac{3}{4} + \frac{3}{4}x + a$, what was to be proved.

$$\begin{aligned} 3C' &= 3C' \\ 4C'' &= 4C'' \\ C''' &= \tfrac{5}{2}B'' + \tfrac{5}{4}D' \\ 6C'''' &= \tfrac{6}{2}B''' + \tfrac{6}{4}D'' \\ 7C^{v} &= \tfrac{7}{2}B'''' + \tfrac{7}{4}D''' \\ 8C^{vi} &= \tfrac{8}{2}B^{v} + \tfrac{8}{4}D'''' \\ \&\text{c.} &= \end{aligned}$$

Therefore $z = \frac{1}{2}y + \frac{1}{2}b + \frac{1}{4}v + \frac{1}{2}d$.

As a matter of fact the first perpendicular Column $= z$, by Hypothesis.

$3C' + 4C'' = \frac{1}{2}$.
$3B' + 4B'' + 5B''' + 6B''''$ &c. $= y$
$B' + B'' + B''' + 6B''''$ &c. $= b$
Therefore there is $4B' + 5B'' + 6B''' + 7B''''$ &c. $= y + b$.
But $4B' = 1$.
Therefore $5B'' + 6B''' + 7B''''$ &c. $= y + b - 1$.
$\frac{5}{2}B'' + \frac{6}{2}B''' + \frac{7}{2}B''''$ &c. $= \frac{1}{2}y + \frac{1}{2}b - \frac{1}{2}$.
Therefore the second perpendicular Column $= \frac{1}{2} + \frac{1}{2}y + \frac{1}{2}b - \frac{1}{2} = \frac{1}{2}y + \frac{1}{2}b$.
Again, $3D' + 4D'' + 5D''' + 6D''''$ &c. $= v$
$2D' + 2D'' + 2D''' + 2D''''$ &c. $= 2d$
Therefore there is $5D' + 6D'' + 7D''' + 8D''''$ &c. $= v + 2d$.
And $\frac{5}{4}D' + \frac{6}{4}D'' + \frac{7}{4}D''' + \frac{8}{4}D''''$ &c. $= \frac{1}{4}v + \frac{1}{2}d$.

Therefore the third perpendicular Column $= \frac{1}{4}v + \frac{1}{2}d$.
Therefore there is $z = \frac{1}{2}y + \frac{1}{2}b + \frac{1}{4}v + \frac{1}{2}d$, what was to be proved.

With the same order there may be written.

$$\begin{array}{lll@{\qquad}lll}
 & 3D' & = 3D' & & 3A' & = 3A' \\
 & 4D'' & = 4D'' & & 4A'' & = 4A'' \\
 & 5D''' & = \frac{1}{2}C'' + \frac{5}{4}B' & & 5A''' & = \frac{1}{2}D'' + \frac{5}{4}C' \\
 & 6D'''' & = \frac{6}{2}C''' + \frac{6}{4}B'' & & 6A'''' & = \frac{6}{2}D''' + \frac{6}{4}C'' \\
 & 7D^{v} & = \frac{7}{2}C'''' + \frac{8}{4}B''' & & 7A^{v} & = \frac{7}{2}D'''' + \frac{7}{4}C''' \\
 & 8D^{vi} & = \frac{8}{2}C^{v} + \frac{8}{4}B'''' & & 8A^{vi} & = \frac{8}{2}D^{v} + \frac{8}{4}C'''' \\
 & \&c. & = & \&c. & & \\
\text{Hence} & v & = \frac{1}{2}z + \frac{1}{2}c + \frac{1}{4}y + \frac{1}{2}b. & \text{And} & x & = \frac{1}{2}v + \frac{1}{2}d + \frac{1}{4}z + \frac{1}{2}c.
\end{array}$$

Which Conclusions indeed are demonstrated in the same manner as above.
But with those four equations solved, there will be elicited

$$y = \frac{45536}{149^2}, \quad z = \frac{38724}{149^2}, \quad v = \frac{37600}{149^2}, \quad x = \frac{33547}{149^2} = \frac{33547}{22201}.$$

Therefore, if B, C, D, A may wish to sell to a certain Spectator R the sums which they individually hope to obtain, fairness will be that the buyer R pay out

$$\begin{array}{llll}
\text{to the one } B & 4 \times \frac{56}{149} + \frac{45536}{22201}p, & \text{to the one } C & 4 \times \frac{36}{149} + \frac{38724}{22201}p, \\
\text{to the one } D & 4 \times \frac{32}{149} + \frac{37600}{22201}p, & \text{to the one } A & 4 \times \frac{25}{149} + \frac{33547}{22201}p.
\end{array}$$

To find the Probabilities which B, C, D, A have, that they must be fined, in a given number of games.

If there are as many as two Games, they will be in this way. $\left.\begin{array}{cc} BA & BA \\ \hline CB & BC \end{array}\right\}$ Whence it is clear B or C to be fined necessarily.

If there will have been three Games, the thing itself is had in this manner.

$$\left.\begin{array}{cccc} BA & BA & BA & BA \\ \hline CB & CB & BC & BC \\ DC & CD & DB & BD \end{array}\right\} \text{Hence it is clear } C, \text{ or } D \text{ or } B \text{ to be fined necessarily.}$$

If indeed there will have been four Games.

$$\left.\begin{array}{cccccc} BA & BA & BA & BA & BA & BA \\ \hline CB & CB & CB & CB & BC & BC \\ DC & DC & CD & CD & DB & DB \\ AD & DA & AC & CA & AD & DA \end{array}\right\} \text{Therefore } A \text{ must be fined in three ways, } D \text{ in two, } C \text{ in one.}$$

And thus with the remaining. From which the Composition of the adjoined Table is manifest of the Probabilities which B, C, D, A have that they be fined, in a given number of games.

	Number of Games	B	C	D	A
i	2	$\frac{1}{2}$	$\frac{1}{2}$		
ii	3	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{2}{4}$	
iii	4		$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$
iiii	5	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{2}{16}$	$\frac{3}{16}$
v	6	$\frac{6}{32}$	$\frac{5}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
vi	7	$\frac{6}{64}$	$\frac{8}{64}$	$\frac{8}{64}$	$\frac{4}{64}$
		&c.			

But let y, z, v, x be the sums of all Probabilities which B, C, D, A have respectively that they should be fined.

Let be written in the same order as in the preceding.

$$\begin{array}{llllll} & B' & = B' & & C' & = C' \\ & B'' & = B'' & & C'' & = C'' \\ & B''' & = \frac{1}{2}A'' + \frac{1}{4}A' & & C''' & = \frac{1}{2}B'' + \frac{1}{4}D' \\ & B'''' & = \frac{1}{2}A''' + \frac{1}{4}A'' & & C'''' & = \frac{1}{2}B''' + \frac{1}{4}D'' \\ & B^v & = \frac{1}{2}A'''' + \frac{1}{4}A''' & & C^v & = \frac{1}{2}B'''' + \frac{1}{4}D''' \\ & B^{vi} & = \frac{1}{2}A^v + \frac{1}{4}A'''' & & C^{vi} & = \frac{1}{2}B^v + \frac{1}{4}D'''' \\ & \text{\&c.} & = & & \text{\&c.} & = \\ \text{Therefore} & y & = \frac{3}{4} + \frac{1}{2}x + \frac{1}{4}x. & \text{Therefore} & z & = \frac{1}{2} + \frac{1}{2}y + \frac{1}{4}v. \\ & & == \frac{3}{4} + \frac{3}{4}x. & & & \end{array}$$

Let be written finally

$$\begin{array}{llllll} & D' & = D' & & A' & == A' \\ & D'' & = D'' & & A'' & = A'' \\ & D''' & = \frac{1}{2}C'' + \frac{1}{4}B' & & A''' & = \frac{1}{2}D'' + \frac{1}{4}C' \\ & D'''' & = \frac{1}{2}C''' + \frac{1}{4}B'' & & A'''' & = \frac{1}{2}D''' + \frac{1}{4}C'' \\ & D^v & = \frac{1}{2}C'''' + \frac{1}{4}B''' & & A^v & = \frac{1}{2}D'''' + \frac{1}{4}C''' \\ & D^{vi} & = \frac{1}{2}C^v + \frac{1}{4}B'''' & & A^{vi} & = \frac{1}{2}D^v + \frac{1}{4}C'''' \\ & \text{\&c.} & = & & \text{\&c.} & = \\ \text{Therefore} & y & = \frac{1}{4} + \frac{1}{2}z + \frac{1}{4}y. & \text{Therefore} & x & = \frac{1}{2}v + \frac{1}{4}z. \end{array}$$

But with those four equations resolved, there will be discovered

$$y = \frac{243}{249} \qquad z = \frac{252}{149} \qquad v = \frac{224}{149} \qquad \& \ x = \frac{175}{149}.$$

Therefore if any Spectator S wishes to sustain all fines, fairness will be that to the one S

B must hand over $\frac{243}{149}p$ $\quad C$ $\frac{252}{149}p$ $\quad D$ $\frac{224}{149}p$ $\quad$ & A $\frac{175}{149}p$.

Therefore with the sums of the probabilities which the Gamesters have individually that they should be fined removed, from the sums of the expectations which the same have if they depart victorious, the lots of them will remain respectively: certainly

B accepts from R $\frac{4\times56}{149}+\frac{45536}{22201}p$
B surrenders to S $\frac{243}{149}p$
Therefore there remains to $B\frac{224}{149}+\frac{9329}{22201}p$
But B had deposited 1 before the game must begin.
Therefore B gains $\frac{75}{149}+\frac{9329}{22201}p.$

C accepts from R $\frac{4\times36}{149}+\frac{38724}{22201}p$
C surrenders to S $\frac{252}{149}p$
Therefore there remains to $C\frac{144}{149}+\frac{1176}{22201}p$
But C had deposited 1.
Therefore C gains $-\frac{5}{149}+\frac{1176}{22201}p.$

D accepts from R $\frac{4\times32}{149}+\frac{37600}{22201}p$
D surrenders to S $\frac{224}{149}p$
Therefore there remains to $D\frac{128}{149}+\frac{4224}{22201}p$
But D had deposited 1.
Therefore D gains $-\frac{21}{149}+\frac{4224}{22201}p.$

A accepts from R $\frac{4\times25}{149}+\frac{33547}{22201}p$
A surrenders to S $\frac{175}{149}p$
Therefore there remains to $A\frac{100}{149}+\frac{7472}{22201}p$
But A had deposited $1+p$, certainly 1 before the game must begin, & p after he had been defeated by B once:
Therefore A gains $-\frac{49}{149}-\frac{14729}{22201}p.$

The gain of $B=+\frac{75}{149}+\frac{9329}{22201}p$
of $C=-\frac{5}{149}+\frac{1176}{22201}p$
of $D=-\frac{21}{149}+\frac{4224}{22201}p$
of $A=-\frac{49}{149}-\frac{14729}{22201}p$

The sum of the gains $=0$.

But the sum of the gains of B & A themselves $=\frac{26}{149}-\frac{5400}{22201}p$;but we had posed B to have defeated the one A once, before the Gamesters undertook the agreements with R & S. Before indeed the game may be started, A had equal lot to expect that he must defeat the one B, and the sum of the gains $\frac{26}{149}-\frac{5400}{22201}p$ should be divided precisely into two parts, thus each gain should be counted as $\frac{13}{149}-\frac{2700}{22201}p$.

Let be put $\frac{13}{149}-\frac{2700}{22201}p=0$, & there will be $p=\frac{1937}{2700}$.

Therefore if the fine p may be to the sum which they deposit individually as 1937 to 2700, A & B gain nothing, lose nothing. Indeed in this case C gains $\frac{1}{225}$, which D loses.

Corollary 1. Spectator R, before the game may begin, will be able to undertake for himself, that the sum 4 concerning which the Gamesters contend, & all fines paid out, if they surrendered to him at the beginning $4+7p$.

Corollary 2. If the skills of the Gamesters are in given ratio, the lots of the Gamesters will be determined by the same calculation.

Corollary 3. If any Series be so constituted, that it may decrease continuously, & any term may have to the preceding any given ratios whatsoever, whether the same or different, that series will be summed precisely. In addition if all terms of this Series are multiplied by the terms of an Arithmetic progression, one by one, the resulting new Series will be summed precisely.

Corollary 4. If there are several parallel Series, so related that any term of each Series to any preceding of the other Series has given ratio, whether the same or different, so that those parallel Series may arrange themselves crosswise by whatever given corresponding law, that Series will be summed exactly. In addition if all terms of these Series are multiplied in order by the terms of an Arithmetic Progression, one by one, the new Series resulting out of this multiplication still will be summed precisely.

Key to the general Problem.

If there are any number of Gamesters *for the sake of an example* Six, B, C, D, E, F, A & the Probabilities which they have that they escape victorious, or that they are fined, in a given number of Games, are denoted respectively $B_{\prime}$, $C_{\prime}$, $D_{\prime}$, $E_{\prime}$, $F_{\prime}$ & $A_{\prime}$; & the Probabilities corresponding to these in a given number of Games with the nearest & with the lesser, by $B_{\prime\prime}$, $C_{\prime\prime}$, $D_{\prime\prime}$, $E_{\prime\prime}$, $F_{\prime\prime}$, $A_{\prime\prime}$; & the Probabilities corresponding to these newest likewise in a given number of Games with the nearest & with the lesser, by $B_{\prime\prime\prime}$, $C_{\prime\prime\prime}$, $D_{\prime\prime\prime}$, $E_{\prime\prime\prime}$, $F_{\prime\prime\prime}$, $A_{\prime\prime\prime}$, & thus in order; there will always be

$$\begin{aligned}
B_{\prime} &= \tfrac{1}{2}A_{\prime\prime} + \tfrac{1}{4}A_{\prime\prime\prime} + \tfrac{1}{8}A_{\prime\prime\prime\prime} + \tfrac{1}{16}A_{v} \\
C_{\prime} &= \tfrac{1}{2}B_{\prime\prime} + \tfrac{1}{4}F_{\prime\prime\prime} + \tfrac{1}{8}E_{\prime\prime\prime\prime} + \tfrac{1}{16}D_{v} \\
D_{\prime} &= \tfrac{1}{2}C_{\prime\prime} + \tfrac{1}{4}B_{\prime\prime\prime} + \tfrac{1}{8}F_{\prime\prime\prime\prime} + \tfrac{1}{16}E_{v} \\
E_{\prime} &= \tfrac{1}{2}D_{\prime\prime} + \tfrac{1}{4}C_{\prime\prime\prime} + \tfrac{1}{8}B_{\prime\prime\prime\prime} + \tfrac{1}{16}F_{v} \\
F_{\prime} &= \tfrac{1}{2}E_{\prime\prime} + \tfrac{1}{4}D_{\prime\prime\prime} + \tfrac{1}{8}C_{\prime\prime\prime\prime} + \tfrac{1}{16}B_{v} \\
A_{\prime} &= \tfrac{1}{2}F_{\prime\prime} + \tfrac{1}{4}E_{\prime\prime\prime} + \tfrac{1}{8}D_{\prime\prime\prime\prime} + \tfrac{1}{16}C_{v}
\end{aligned}$$

And the retrogression in succession may happen always to as many letters less by two as many as there are Gamesters, and the letter A may always be omitted, with the first equation excepted, where the letter A occupies all terms except the first.