PROPOSITIONS XL & XLI
ON THE PROBLEM OF POINTS

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EXTRACTED FROM
ESSAY D'ANALYSE SUR LES JEUX DE HAZARD
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PROBLEM VI
PROPOSITION XL.

To determine generally the divisions which one must make between several players who play at an equal game in several parts.

187. Although this Problem is least difficult of those which one can intend on this matter when it is limited to two Players, & when the conditions of the game are the same for the Players, it was not permitted to exercise a long time, in that which appears with pleasure, two illustrious geometers, Messrs. Fermat & Pascal. The one employed the analytic method in order to come to the end; this view seems to be here most natural & most easy; but it has the fault to be of an excessive length, because one can not find the solution of the slightly complicated case unless one has examined all those which are less, by commencing with the simplest. Thus, for example, to find by this view the lot of three Players Pierre, Paul & Jacques, by supposing that Pierre plays for a point, Paul for two, & Jacques for three, it would be necessary to examine ⟨1°⟩ what would be their lot, if Pierre played for one point, Paul played similarly only for one point, & Jacques either for one, or for two, or for three points; 2° what would be their lot if Pierre played for two points, Paul & Jacques played similarly for two points, that which would fall back next into the preceding case.

The method of M. Fermat is more learned, & demands more skill in its application. He has employed it only to determine the division between two Players. M. Pascal does not believe that it can be extended to a greater number. I will show that the method of M. Fermat solves the problem of the division in a very general manner. But in order to make it understood, & to make known the difficulties that M. Pascal found, I believe it would not be done better than to bring back here the Letter of 24 August 1654 which is entirely on this subject. It is addressed to M. Fermat, & is found in this posthumous Works printed *in folio* at Toulouse: We will see the explication of the method of M. Fermat for two Players, & the doubts of M. Pascal on this method when one wishes to apply it to a greater number. I will give next the solution of the difficulties of M. Pascal, & I will apply this method to some Examples, which will demonstrate the universality of it.

[*Letter of Mister Pascal to Mister de Fermat.*]

The respect that we have for the reputation & for the memory of M. Pascal, does not permit me to remark here in detail all the faults of reasoning which are in this Letter; it

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will suffice us to a caution that the cause of his error is to have no regard to the diverse arrangements of the letters.

In order to prove that of the twenty-seven different positions that the three dice can have, there are seventeen which make Pierre win, & five which make win each of the two other players to whom there are lacking two points; here is how it seems to me that we must reason.

The three Players are obligated to play three games, but with the condition that if Pierre to whom there is lacking only one point, wins it before either one or the other of the other Players has won two points, he will win the game; & that he loses it if one or the other Player to whom there is lacking two points, can take them before Pierre has taken one of them. It is evident that this assumption returns precisely to that of the Problem. Now according to this assumption we will find that of the twenty-seven positions of the three dice, there are seventeen which will make Pierre win, five which will make Paul win, & five which will make Jacques win, thus it will appear according to the following Table.

TABLE.

		Pierre		Paul	Jacques
aaa	abc	bab	cac	bba	cca
aab	aca	bac	cba	bbb	ccc
aac	acb	bca		bbc	ccb
aba	acc	caa		bcb	cbc
abb	baa	cab		cbb	bcc

REMARK I.

188. The general rule, is to examine in how many trials at least the game must necessarily end; to take as many of the dice as there are of these trials, & to give to these dice as many faces as there are Players; next the question is no more than to determine among all the possible dispositions of these dice, which are those which are advantageous & contrary to each of the Players, which we will find always easily by *art. 29 & 42*.

Thus, for example, in supposing that Pierre plays for one point, Paul for two, & Jacques for three, if we wish to know the lot of each of these three Players, it would be necessary to imagine revealed four dice marked with three points each, for example with one 1, with one 2 & with one 3; to seek next by our rules of combinations in how many ways it can be found one ace which precedes either two 2, or three 3, & in how many way two 2 or three 3 can precede the ace, which will give the following Table. Whence it seems that out of eighty casts there are fifty-seven for Pierre, eighteen for Paul, & six for Jacques.

We can resolve the preceding Problem in a more abridged manner, by making the reasoning which follows.

I remark that we would not do harm to any of these Players, if we obliged them to play three trials with these conditions. 1° That if Pierre won a cast before Paul has won two of them, it would make sense to have won the game. 2° That if Paul won two casts before Pierre had won one of them, Paul would win. 3° That Jacques would have won if he won the three casts. 4° That if of the three casts Paul won one of them, & Jacques two, the Players would separate themselves by withdrawing to each of them their wager.

In order to calculate all this easily, we can, as before, imagine three dice which have each three faces, that on one is an ace, on the other a 2, on the third a 3, & to suppose that out of the twenty-seven casts that we can bring forth with these three dice, all those where there is found an ace which precedes two 2, will be favorable to Pierre, & that all those where two 2 will precede the ace will be for Paul. We will find by art. 29 & 42, that there are eighteen casts which give A to Pierre, by supposing that A expresses all the money of

TABLE

	Pierre	Paul	Jacques
$1,1,1,1$	1	0	0
$1,1,1,2$	4	0	0
$1,1,1,3$	4	0	0
$1,1,2,2$	5	1	0
$1,1,3,3$	6	0	0
$1,1,2,3$	12	0	0
$1,2,2,3$	8	4	0
$1,2,3,3$	12	0	0
$1,2,2,2$	2	2	0
$1,3,3,3$	3	0	1
$2,2,2,2$	0	1	0
$2,2,2,3$	0	4	0
$2,2,3,3$	0	6	0
$2,3,3,3$	0	0	4
$3,3,3,3$	0	0	1

the game, namely $1,1,1$, which happens in one sole way; $1,1,2$; $1,1,3$; $1,3,3$, each in three ways; $1,2,3$, which happens in six ways; & these two here $1,2,2$; $2,1,2$. That there are five favorable to Paul, namely $2,2,1$; $2,2,2$, & $2,2,3$ in three ways; & one sole cast which gives A to Jacques. We will find finally that here are three casts which give $\frac{1}{3}A$ to each of the Players, namely $2,3,3$.

It is easy to note what are the cases where one can abridge in this way the general method.

REMARK II.

When there are many Players for whom there are lacking several points, the method which proceeds by combinations & changes of order, is so lengthy, & falls into such great detail than that which proceeds by analysis, because a like cast of the dice can be favorable to different Players, it seems that we cannot dispense considering that which furnishes each different cast of dice in particular, & this examination can be only so long & so inconvenient; but the method of M. Fermat, beyond several advantages which it has over that of M. Pascal, has the one of resolving in a short & simple way the Problem in question, when the question is only of two Players. Here is the solution which it furnishes.

PROBLEM
PROPOSITION XLI.

Let p be the number of points which are lacking to Pierre, q the number of points which are lacking to Paul. We demand a formula which expresses the lot of the Players.

SOLUTION.

190. Let $p+q-1=m$, the lot of Pierre will be expressed by a fraction of which the denominator will be 2 raised to the exponent m, & of which the numerator will be composed of as many terms of this series $1+m+\frac{m.m-1}{1.2}+\frac{m.m-1.m-2}{1.2.3}+\frac{m.m-1.m-2.m-3}{1.2.3.4}+$&c. as q expresses units, the lot of Paul will be the complement of unity.

If we suppose that the number of chances that Pierre has to win each point, or if we wish that his lot be to that of Paul as a to b; we will have likewise the lot of Pierre by multiplying the terms of this series which are the coefficients of the power m, by the powers of a & of b which correspond to them (*art. 27*); thus the preceding series becomes $a^m b^0 + ma^{m-1}b + \frac{m.m-1}{1.2}a^{m-2}bb + \frac{m.m-1.m-2}{1.2.3}a^{m-3}b^3 + \frac{m.m-1.m-2.m-3}{1.2.3.4}a^{m-4}b^4 +$&c. which it is necessary to continue to the number of terms expressed by q, & to divide by $(a+b)^m$. The formula which designates the lot of Paul is $1 \times b^m a^0 + mb^{m-1}a + \frac{m.m-1}{1.2}b^{m-2}aa + \frac{m.m-1.m-2}{1.2.3}b^{m-3}a^3 + \frac{m.m-1.m-2.m-3}{1.2.3.4}b^{m-4}a^4 +$&c. continued to the number of terms expressed by p, & divided by $(a+b)^m$.

For example, if Pierre plays for five points, & Paul for three, the last formula gives the lot of Paul $= b^7 + 7b^{7-1}a + 21b^{7-2}aa + 35b^{7-3}a^3 + 35b^{7-4}a^4$, the whole divided by $(a+b)^7$.

DEMONSTRATION.

The demonstration of this formula is founded, 1°. On this that the game must necessarily end in as many trials less one as there are units in the sum of the points which are lacking of both.

2°. That supposing m dice of which each has two faces, one white, the other black, there are $\frac{1 \times b^m}{(a+b)^m}$ of them for which casting them at random are found m white faces, $\frac{m \times b^{m-1}a}{(a+b)^m}$ for which are found $m-1$ white faces & one black, $\frac{m.m-1}{1.2} \times \frac{b^{m-2}a^2}{(a+b)^m}$ for which are found $m-2$ white faces and two black, &c. thus as it is demonstrated, *art.* 27.

3°. That it is the same thing to wager that Pierre will win p points in m trials, or to wager that casting m dice at random there will be brought forth p white faces.

ANOTHER FORMULA.

191. The Analysis has again furnished me another formula. Supposing the same denominations as above, I find the lot of Pierre $= \frac{1 \times a^p b^0}{(a+b)^p} + \frac{p \times a^p b^1}{(a+b)^{p+1}} + \frac{p.p+1 \times a^p b^2}{1.2.(a+b)^{p+2}} + \frac{p.p+1.p+2 \times a^p b^3}{1.2.3.(a+b)^{p+3}} + \frac{p.p+1.p+2.p+3 \times a^p b^4}{1.2.3.4.(a+b)^{p+4}} +$&c. And likewise of Paul $= \frac{1 \times b^q a^0}{(a+b)^q} + \frac{q \times b^q a^1}{(a+b)^{q+1}} + \frac{q.q+1 \times b^q a^2}{1.2.(a+b)^{q+2}} + \frac{q.q+1.q+2 \times b^q a^3}{1.2.3.(a+b)^{q+3}} + \frac{q.q+1.q+2.q+3 \times b^q a^4}{1.2.3.4.(a+b)^{q+4}} +$&c.

The formula which expresses the lot of Pierre will have as many terms as there are units in q, & that which expresses the lot of Paul as many terms as there are units in p.

In order to demonstrate this second formula more easily, I am going to make application of it in the example above where Pierre plays for five points & Paul for three. There are

$\frac{1 \times b^3}{(a+b)^q}$	in order that Paul win his three trials in sequence	$\frac{1 \times b^3}{(a+b)^3}$
$\frac{4 \times b^3 a}{(a+b)^{q+1}}$	in order that Paul win in four trials, of which it is necessary to reduce 1 for the preceding case, leaving	$\frac{3 \times b^3 a}{(a+b)^4}$
$\frac{10 \times b^3 aa}{(a+b)^{q+2}}$	in order that Paul win in five trials, of which it is necessary to reduce $1+3=4$ for the two preceding cases, leaving	$\frac{6 \times b^3 aa}{(a+b)^5}$
$\frac{20 \times b^3 a^3}{(a+b)^{q+3}}$	in order that Paul win in six trials, of which it is necessary to reduce $1+3+6=10$ for the preceding cases, leaving	$\frac{10 \times b^3 a^3}{(a+b)^6}$
$\frac{35 \times b^3 a^4}{(a+b)^{q+4}}$	in order that Paul win in seven trials, of which it is necessary to reduce $1+3+6+10=20$ for the preceding cases, leaving	$\frac{15 \times b^3 a^4}{(a+b)^7}$

So that generally the coefficients of the series which expresses the lot of the Player to whom is lacking q, are the numbers of the horizontal band, *art. 1*, of which the heading is expressed by q; the coefficients of the series which expresses the lot of the Player to whom is lacking p points, are the numbers of the horizontal band of which the heading is p.

It is easy to observe, that the numbers 1, 4, 10, 20, 35, &c. are found by Proposition 14 & *art. 41*.2°. That if we put to like determination all the terms of the second formula, where the divisor $a + b$ is found raised to different powers, we will rediscover the first formula. Thus in the example above multiplying $1b^3$ by $(a + b)^4$, & $3b^3a$ by $(a + b)^3$, & $6b^3aa$ by $(a+b)^2$, & $10b^3a^3$ by $(a+b)$, & $15b^3a^4$ by $(a+b)^0$; we will have the expression of the lot of Paul

$$\frac{1 \times b^7a^0 + 7b^{7-1}a^1 + 21b^{7-2}a^2 + 35b^{7-3}a^3 + 35b^{7-4}a^4}{(a + b)^7}$$

conforming to the first formula.

EXAMPLES.

192. *Pierre plays for five points, & Paul for six points*: the lot of Pierre is 319 against 193.

Pierre plays for four points, & Paul for six points: the lot of Pierre is 191 against 65, which is a little less than 3 against 1.

Pierre plays for three points, & Paul for six points: the lot of Pierre is 219 against 37, which is a little less than 6 against 1.

Pierre plays for two points, & Paul for six points: the lot of Pierre is 15 against 1.

Pierre plays for one point, & Paul for six points: the lot of Pierre is 63 against 1.

REMARK I.

193. The lot of Pierre when $\frac{p}{q} = r$, being $= A$, it does not follow that c & d expressing in another game the number of points which would be lacking to Pierre & to Paul, & the fraction $\frac{c}{d}$ being again $= r$, the lot of Pierre is always the same, or $= A$ even in the case of $a = b$. This remark is quite important for the Players. Because we can easily believe that the lot of Pierre is the same when he plays for one point & Paul for two, or when he plays for two points & Paul for four, or when he plays for three points & Paul for six. We will find by the formulas, & we can also recognize it by simple reasoning that the lot of Pierre is better in the last case than in the preceding; & generally $\frac{c}{d}$ being $= \frac{p}{q}$, the lot of Pierre will be always so much better, when c & d will designate greater numbers, in ratio to p & to q; in such a way that if a Player can give eight points to sixteen at Billiards to another Player: we can not conclude that he can without disadvantage give to him four of eight.

REMARK II.

194. It would be much wished for that we could find for three Players, four Players, &c. some formulas parallel to the preceding for two Players. which determined all in one stroke their lot, by supposing their strengths equals or unequals, because the Problem is not more difficult in one fashion than in the other; but there is place to believe that this research is extremely difficult, & there is the appearance that we can add nothing to that which we have given above in the Remark, *art. 188*.