PROBLEME RESOLU PAR L'AUTEUR
DE L'ANALYSE SUR LES JEUX DE HAZARD

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To find the sum of any series of figurate numbers, of which all the terms are raised to any exponent, either whether these terms are taken in sequence, or interrupted by some equal distances.

By figurate numbers I intend not only those which compose the arithmetic triangle of Mr. Pascal, of which the first order or horizontal rank is composed of units; the second,
of natural numbers   1, 2, 3, 4, 5, 5, 7, &c.
the third, of triangular numbers   1, 3, 6, 10, 15, 21, 28, &c.
but more generally of the sequences of numbers, of which the first

order is	r,	r,	r,	r,	r,	r,	&c.
the second,	s,	$r+s$,	$2r+s$,	$3r+s$,	$4r+s$,	$5r+s$,	&c.
the third,	s,	$r+2s$,	$3r+3s$,	$6r+4s$,	$10r+5s$,	$15r+6s$,	&c.
the fourth,	s,	$r+3s$,	$4r+6s$,	$10r+10s$,	$20r+15s$,	$35r+21s$,	&c.

& thus of the rest, so that the second order being composed of numbers in any arithmetic progression, each term is equal to the one which is immediately above, & to the one which is to the left.

SOLUTION

Let p be the number of terms of which one wishes to see the sum, a, the first term of the series; b, the second; c, the third; d, the fourth; e, the fifth; f, the sixth, &c. Let also $b - a = C$, $c - \overline{a + 2C} = D$, $d - \overline{a + 3C + 3D} = E$, $e - \overline{a + 4C + 6D + 4E} = F$, $f - \overline{a + 5C + 10D + 10E + 5F} = G$, &c. the coefficients of the numbers a, C, D, E, F, G, being always the same as those which are found by the formation of the powers, or by the perpendicular bands of the arithmetic triangle. Let next m be the exponent of the terms of the series, n the number of the order, one will have the sought sum $= ap + \frac{p \cdot p - 1}{1.2}C + \frac{p \cdot p - 1 \cdot p - 2}{1.2.3}D + \frac{p \cdot p - 1 \cdot p - 2 \cdot p - 3}{1.2.3.4}E + \frac{p \cdot p - 1 \cdot p - 2 \cdot p - 3 \cdot p - 4}{1.2.3.4.5}F + \frac{p \cdot p - 1 \cdot p - 2 \cdot p - 3 \cdot p - 4 \cdot p - 5}{1.2.3.4.5.6}G+$ &c.
It is necessary to take as many terms of this formula as $m \times \overline{n - 1} + 1$ expresses units. Suppose, for example, that one demands the sum of the first one hundred numbers of the third order, raised to the square, by supposing $r = 2$ & $s = 1$, one will have
$C = b - a = 15D = c - \overline{a + 2C} = 81 - 31 = 50$
$E = d - \overline{a + 3C + 3D} = 256 - 196 = 60$
$F = e - \overline{a + 4C + 6D + 4E} = 625 - 601 = 24$. One will have also $m = 2$, $n = 3$ & consequently $m \times \overline{n - 1} + 1 = 5$. Adding therefore the first five terms of the general formula, & substituting for C, D, E, F their values, one will have 2050333330 for the sought value.

If one wished to have a formula for this particular case, one would draw it easily from the general formula; & one would find $\frac{6p^5+15p^4+10p^3-p}{30}$. One will have likewise the formula of the triangular numbers raised to the cube.

$$\frac{15p^7 + 105p^6 + 273p^5 + 315p^4 + 140p^3 - 8p}{2.3.4.5.6.7}$$

& that of the pyramidal numbers, taken two by two, 1, 10, 35, 84, 165, &c. raised to the square.

$$\frac{1280p^7 + 4480p^6 + 3584p^5 - 2240p^4 - 2800p^3 + 280pp + 456p}{1.2.3.4.5.6.7}$$

This Problem has, as one sees, all the extent & all the universality possible, & leaves nothing to desire on this matter, which has yet been treated by no other person that I know. Perhaps nonetheless that it will not be completely useless to join another method, in order *to find the sum of the squares, of the cubes, square squares, & generally of all the powers of the natural numbers* 1, 2, 3, 4, 5, 6, &c. This method is quite limited in comparison with the preceding; but it has the advantage to be quite simple, & to carry with it its demonstration.

Let B be the sum of a series of as many natural numbers raised to the square as there are units in p, & A the sum of a series of as many natural numbers as there are units in p, one has by propositions 10 & 11 of the *Essai d'Analyse sur les Jeux de hazard* $\frac{B}{2} + \frac{A}{2} = \frac{p \times p+1 \times p+2}{1.2.3} = \frac{p^3+3pp+2p}{6}$ therefore $\frac{B}{2} = \frac{p^3+3pp+2p}{6} - \frac{A}{2}$ & by substituting for A its value $\frac{pp+p}{2}$, $B = \frac{2p^3+3pp+p}{6}$ formula of the square.

Let now C be the sum of a series of as many natural numbers raised to the cube, as there are units in p, one has by the same propositions 10 & 11 $\frac{C}{6} + \frac{3B}{6} + \frac{2A}{6} = \frac{p^4+6p^3+11pp+6p}{1.2.3.4}$ or $C = \frac{p^4+6p^3+11pp+6p}{4} - 3B - 2A$, & by substituting for B its value $\frac{2p^3+3pp+p}{6}$ & for A its value $\frac{pp+p}{2}$ one has $C = \frac{\overline{p+1}^2 \times pp}{4}$ formula of the cubes.

Let next D be the sum of a series of a series of as many natural numbers raised to the fourth power, as there are units in p, one will have

$$\frac{D + 6C + 11B + 6A}{24} = \frac{p \times p+1 \times p+2 \times p+3 \times p+4}{1.2.3.4.5}$$

and by substituting for C its value $\frac{\overline{p+1}^2 \times pp}{4}$ for B its value $\frac{2p^3+3pp+p}{6}$ for A its value $\frac{pp+p}{2}$ one will find $\frac{6p^5+15p^4+10p^3-p}{30}$ for the formula of the square square, & thus in sequence for all the other powers.

COROLLARY

One is able by this last method to find a general formula in order to have the sum of the terms of the second order or horizontal rank s, $r + s$, $2r + s$, $3r + s$, $4r + s$, $5r + s$, $6r + s$, &c. raised to any exponent.

Let n be the dimension to which are raised all the terms of the series, $p + 1$ the number of terms of which one wishes to have the series. The sought sum will be $p + 1 \times s^n + n \times \frac{pp+p}{2} \times r \times s^{n-1} \times \frac{n \times n-1}{1.2} \times \frac{2p^3+3pp+p}{6} \times r^2 \times s^{n-2} + \frac{n \times n-1 \times n-2}{1.2.3} \times \frac{\overline{p+1}^2 \times pp}{4} \times r^3 \times s^{n-3} + \frac{n \times n-1 \times n-2 \times n-3}{1.2.3.4} \times \frac{6p^5+15p^4+10p^3-p}{30} \times r^4 \times s^{n-4} +$ &c. observing that $\frac{pp+p}{2}$ is the sum of the units, $\frac{2p^3+3pp+p^2}{6}$ the sum of the squares, $\frac{\overline{p+1}^2 \times pp}{4}$ the sum of the cubes, &c.

Problem proposed to the Geometers

There was drawn last year at Paris, a Lottery known under the name of the Lottery of Lorraine, of which the Public, has not had subject to be content. When it was published I myself perceived first that one incurred risk of being the dupe in it, & that one would have owed to oblige the Director of the Lottery to give good & sufficient caution, since according to the conditions to which he was obligated, it was possible that he had to render 424950 livres beyond the 500000 livres that he had received. I see in gross, that his part was not good; & beyond I suspected that one had design to grab the money of the Public, that which has happened.

I went no further then, & I postponed to a time where I would have more leisure, to examine at base the disadvantage of the one who held the Lottery. Having found it recently, I have believed that it would not be useless to propose research on it to the Geometers. Those who have the most esteem for Algebra & Analysis, do not know at all enough how much use it has with respect to the things of civil life. It is good, it seems to me, to give here a new proof; & at the same time, to make known to the Magistrates who would have to decide on a matter of the nature of that here, which is of their competence, that the Geometers are the only ones from whom they are able to receive certain decisions.

Rules of the Lottery

The tickets were six sols, & there were one million. For the 500000 livres that he received from the Public the one who held the Lottery, he rendered to it 425000 livres in twenty thousand lots. Two conditions would make the novelty & the singularity of this Lottery.

1° The one who held the Lottery, in order to compensate the Public of the 75000 livres that he retained, would be obliged to render 25 livres to each of those who having taken 50 tickets in sequence, would have no lot in their 50 tickets.

2° Here is in what manner the Lottery was held. All the tickets or numbers were in a box, & the black tickets in another. One drew at the same time a black ticket & a number; & after one had written what number has one such lot, one cast outside the black ticket, & one returned the number into the box with the numbers; so that in this manner of drawing the lots, one same number would be able to win many lots, or even to will them all.

PROBLEM

In supposing that all those who set in the Lottery took either 50 tickets, or 100, or 150, &c. (this assumption appears completely receivable;) *One demands what is the advantage, or the disadvantage of the one who holds the Lottery?* It is easy to observe, 1° That the Director of the Lottery will win 75000 livres, if all those who have set in the Lottery have a lot in each fifty tickets. 2° That he will lose 424950 livres if only one of all those who have set in the Lottery carry away all the lots. 3° That he will neither lose, nor will win, if three thousand persons only have no lots at all in their fifty tickets. Whence it follows that this Lottery is a kind of game of chance, where the one who holds the Lottery is able to lose & to win. The solution of this Problem is hidden under this Anagram $4a$, $5e$, 51, 130, $3u$, $2l$, $2n$, $2p$, $4s$, 32, c, d, m, r, of which I will give the explication when one will wish it.

SOLUTION OF THE PROBLEM

Inserted into the Journal of Sçcavans, year 1711 page 186. sent by N. Bernoulli of Basel.

It has been some months that the Author of the Analysis on the Games of chance makes me part of this Problem, on the subject of the Lottery of Lorraine. I sent to him the solution of it the 26 of the month of February last, which being found quite correct, he came to pray to me to render it public; this which I am unable to refuse to him, as much more as he

summons me, as many have worked until now uselessly. Here is therefore an Extract of the Letter that I have had the honor to write to him.

"The Problem that you have had design to propose to the Geometers, has no difficulty; here is how I have conceived the thing. The concern is to find how much the condition to render their 25 livres to those who have taken 50 tickets would have won no lot in their 50 tickets, give the advantage or the disadvantage to the one who holds the Lottery; this which is the same thing as if one wished to seek the lot of the one who undertook to bring forth with 20000 dice with 1000000 faces, of which 50 alone are marked with points, in a single trial at least one of the marked faces. Now the number of cases that do not arrive is 999950^{20000}" (such would be the number of all the different trials that 20000 dice are able to give, of which each has 999950 faces) "& the number of all the cases 1000000^{20000}, whence it follows that this condition of rendering his money to each who wins no lot in his 50 tickets, is worth $\left(\frac{999950}{1000000}\right)^{20000} \times 25$ livres this which makes in all $\left(\frac{999950}{1000000}\right)^{20000} \times 500000$ livres $=$ (this which is found by logarithms) 184064 livres & around 10 sols. Therefore the disadvantage of the Banker who retains only 75000 livres will be $109064\frac{1}{2}$ livres. One will be able by this same method, & in two words, resolve Proposition 44 of your Book."