PROBLEM I
ON THE GAME OF THE SMALL QUOIT
OR OF THE BOWL

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ESSAY D'ANALYSE SUR LES JEUX DE HAZARD
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PROPOSITION XLIII

Pierre plays with a certain number of quoits or of bowl m, Paul with a certain number of quoits or bowls n, the force of Pierre is to that of Paul as a is to b, that which signifies that Pierre & Paul playing each with one bowl or one Quoit, there would be odds a against b that Pierre would approach more of the end than Paul. One supposes that he lacks to Pierre in order to win a certain number of points p, & to Paul a certain number of points q. One demands the lot of the two Players.

SOLUTION.

195 Let be supposed first that there is lacking one point to Pierre, & one, or two, or three, or four, or five &c. points to Paul. Let also in the first case the lot of Paul be called A, in the 2nd B, in the third C, in the fourth D, in the fifth E, &c.

Let further the lot of Paul be expressed by the letters F, G, H, I, L, &c. when there lacks two points to Pierre in order to win, & when there lacks of them to Paul either one, or two, or three, or four, or five, &c.

Let further the lot of Paul be expressed by the letters M, N, O, P, Q, &c. when there lacks three points to Pierre in order to win, & when there lacks of them to Paul either one, or two, or three, or four, or five, &c.

Let further the lot of Paul be expressed by the letters R, S, T, V, X, &c. when there lacks four points to Pierre in order to win, & when there lacks of them to Paul either one, or two, or three, or four, or five, &c.

And thus in sequence. Let also in order to shorten $bn + am = g$, I find the formulas here joined.

By examining the order of these formulas, & by browsing them with the eyes, nothing is so easy as to observe the last of them to continue to infinity: Here is the demonstration of it.

In order to make understood more easily, I am going to apply it to an example.

Let be supposed that the force of Pierre is to the force of Paul, as two is to three, that there lacks four points to Pierre, & five points to Paul; that Pierre plays with four bowls, & Paul with five. In order to find the lot of Paul, I reduce the Problem to this one here. Let be set into a purse two white tokens, two black tokens, two gray tokens, two brown tokens. *Likewise*, three reds, three oranges, three yellows, three greens, three blues. I seek what is my lot, 1 °, in order to draw five tokens of these last colors before drawing any one of

them from the first four. 2 ° In order to draw four of these last five colors before drawing from the first four of them. 3 ° In order to draw three of the last five colors before drawing from the first four of them. 4 ° In order to draw two of the last five colors before drawing from the first four of them. 5 ° In order to draw one of the last five colors before drawing from the first four of them. 6 ° In order to draw one token from the first four colors before drawing from the last five of them. 7 ° In order to draw two tokens from the first four colors before drawing from the last five of them. 8 ° In order to draw three tokens from the first four colors before drawing from the last five of them.

The first case gives a win to Paul, the second gives to him R because it puts him in the situation of winning only for one point, & Pierre for four. The third case gives to him S, because it puts him in the situation to play for two points & Pierre for four. For the same reasons the fourth case gives to him T, & the fifth gives to him V. One will note further that the sixth gives to him Q, because it puts him into the situation to play for five points, & Pierre only for three; & likewise the seventh gives to him L, & the eighth E. Now in our example, when I draw first one token, I see that I have $\frac{15}{23}$ for that which is one token of the last five colors: upon which if it is a red, for example, one takes off the red tokens, & there is $\frac{12}{20}$ in order that I draw a token from the last four colors remaining, rather than the first four. Upon which, supposing that one has drawn the 2$^{\text{nd}}$ time an orange, I have $\frac{9}{17}$ in order to draw a token from the last 3 colors remaining yellows, greens & blues before drawing from the first 4 colors of them, & thus in sequence: whence I draw that the lot of Paul, in order to win the five points which are lacking to him, is $\frac{15.12.9.6.3}{23.20.17.14.11}$. One will find by similar reasoning & by our rules of combinations that I have $\frac{15.12.9.6\times8}{23.20.17.14\times11}$ in order that after having drawn four tokens from the last five colors, I draw a token from the first four, in which case Paul winning only three points, has the lot expressed by S, because he finds himself in the situation to play for two points, Pierre playing for four points. And thus in sequence by continuing, one will find that in the proposed example the lot of Paul is

$$\frac{15.12.9.6.3}{23.20.17.14.11} + \frac{15.12.9.6\times8}{23.20.17.14.11}R + \frac{15.12.9\times8}{23.20.17.14}S + \frac{15.12\times8}{23.20.17}T$$
$$+ \frac{8\times15}{23.21}Q + \frac{8.6\times15}{23.21.19}L + \frac{8.6.4\times15}{23.21.19.17}E,$$

conforming to the formula.

So that the foundation of the method is to imagine, 1 °, as many different colors of tokens for Paul as there are units in n, & as many different colors of tokens for Pierre as there are units in m. 2 ° As many tokens of each of the colors favorable to Paul as there are units in b, & as many tokens of each of the colors favorable to Pierre as there are units in a. This idea being well conceived, the rest is no more than calculation.

COROLLARY I.

196 One sees by examining the formula above that they are shortened when m and n express the unit, all the terms where there enters $m - 1$ & $n - 1$ becoming then zero; & that m being $= n = 1$, the formulas become under another form the same as in art. 190, this which gives a new demonstration of it.

COROLLARY II.

197. The method of which one comes to serve oneself for the case of two Players who play with any number whatever of quoits or bowls, can be employed for an indeterminate number of Players: Here is an example of it in the following Problem.

PROBLEM II.

Pierre, Paul & Jacques play at the bowl each for himself. The forces of these three Players are respectively as 3, 2, 1, there lacks three points to Pierre in order to win, two points to Paul, & one point to Jacques. Pierre plays with three balls, Paul with two, Jacques with one. One demands what is their lot, & what expectation each of the Players has to win.

SOLUTION.

198. If one imagines a purse in which there are three white tokens, three gray, three Isabelles favorable to Pierre. Two black tokens & two brown tokens favorable to Paul. And finally one blue favorable to Jacques. One will find, by making the same reasonings as in the preceding Problem, the lots of the three Players as the three numbers 60143391, 22555106, 14916559.

So that if the money of the game is one pistole or 10 livres & if it is necessary to make reparation between the three Players with respect to the expectation that each has to win

Pierre will have	6 liv.	3 f.	3 d.
Paul	2	6	2
Jacques	1	10	7

COROLLARY

199. If the forces of each of the three Players were equals, all the rest remaining as above, one would have their lots as the three numbers 1581, 1330, 1589; so that under this assumption

Pierre will have	3 liv.	10 f.	3 d. $\frac{1}{5}$
Paul	2	19	$1\frac{1}{3}$
Jacques	3	10	$7\frac{7}{15}$

It had assuredly been quite difficult to divine that the lot of Pierre must find itself better than the one of Paul, & less than the one of Jacques; but less only by the eleven hundred twenty-fifth part of unity.

If there lacks three points to Pierre, two points to Paul, and one point to Jacques, these three Players playing each with one ball, & their forces as the numbers 3, 2, 1, respectively, their lots are among them as the numbers 9, 11, 16. If they play with two bowls each, their lots are as these three numbers 169, 170, 201. If they have three bowls each, their lots are as these three numbers 825, 742, 833. If they have each four bowls, their lots are as these three numbers 1728, 1490, 1633.

PROBLEM III.

Pierre plays for one point, Paul for two, the force of Pierre is double that of Paul, Pierre plays with m bowls. One demands what must be the value of n, that is to say with how many bowls Paul must play in order that the game be equal.

SOLUTION.

200. One has by the formula, art. 195, $\frac{bbn^2 - bbn + b^2n^2 \times \frac{am}{bn+am}}{bn+am \times bn+am-b} = \frac{1}{2}$, or

$$n^3 - 2nnm - 12nmm - 8m^3 + nn + 4mm + 8mn = 0,$$

by substituting for a, 2; & for b, 1. Now if in this equality on supposes for m successively 1, 2, 3, 4, 5, 6, &c. that is to say, that Pierre plays with one, or two, or three, or four, or five, or six bowls, &c. one will find, by resolving the equality that Paul must not have less than three, eight, thirteen, eighteen, twenty-eight, &c.

And if one supposed the force of Pierre triple of that of Paul, one would have the equality

$$n^3 - 3nnm - 27mmn - 27m^3 + nn + 12nm + 9mm = 0,$$

which makes known that if Pierre plays with one bowl it is necessary that Paul play with six, & that if Pierre plays with two bowls, it is necessary that Paul play with thirteen, &c.

This will be the same thing in order to find the forces, the ratio of the bowls being given. But one will have even for the very simple cases, some extremely composite equalities, & always so much more as there lack more points to the Players.

COROLLARY.

201. The ratio of the points which are lacking to each of the Players, the ratio of the forces, the number of quoits or of bowls, any two of these three things being given, one will determine the third.

REMARK.

202. It happens often in the little quoit, in the bowl, in the billiard & other Games, that when there are three Players, there is one of them who carries the two others. One says that Paul carries Pierre & Jacques in the little quoit, when he plays two trials, & when he has two quoits against the others each one, & when these two here are a society of interest against Paul. One can imagine on this idea many curious Problems, of which one will find the solution by that which precedes. Here is one of them which will serve as rule for the others.

PROBLEM IV.

Pierre, Paul & Jacques play each for three points at the little quoit, Pierre & Jacques are together a coalition, & they play each with one quoit. Paul carries the two, & plays with two quoits. The forces of the three Players are respectively as the numbers 3, 2, 1. One demands if Paul has the advantage, or if it is the coalition of Pierre & Jacques.

SOLUTION.

203. Let M be the lot of Paul when he plays for one point & Pierre for three. N his lot when he plays for three points & Pierre for three. C his lot when he plays for three points, & Pierre only for one. H his lot when he plays for three points, & Pierre for two. Let also be imagined that there is in a purse three red tokens favorable to Pierre, one white token favorable to Jacques, two black tokens & two brown tokens favorable to Paul. One will find the lot of Paul

$$= \frac{4}{8} \times \frac{2}{6} \times M + \frac{4}{8} \times \frac{4}{6} \times N + \frac{3}{8} \times \frac{1}{5} \times C + \frac{1}{8} \times \frac{3}{7} \times C + \frac{3}{8} \times \frac{4}{5} \times H + \frac{1}{8} \times \frac{4}{7} \times H.$$

One will find also $M = \frac{4013}{4900}$, $N = \frac{3309}{4900}$, $C = \frac{7}{36}$, $H = \frac{151}{420}$, this which gives the lot sought of Paul$= \frac{2823}{7350}$; so that if the money of the game is one pistole, & if one wishes to divide with respect to the advantage or disadvantage of each Player, it is necessary that Paul take 5 liv. 4 f. $\frac{16}{49}$ d. & the two associates 4 liv. 15 f. $11\frac{33}{49}$ f.

COROLLARY I.

204. One can believe that Pierre & Jacques on one part & Paul on the other, playing for a determined number of points, it is indifferent that they play with any number of bowls as it be, provided that Pierre & Jacques having each an equal number of bowls, Paul has as many of them as the two others together. But this is not so; because, for example, the advantage of Paul which is $\frac{26}{1365}$ when the game being in two points, Paul has two bowls against the others each one, is only $\frac{8}{1365}$, when the game being again at two points, Paul plays with four bowls against Pierre & Jacques each two.

COROLLARY II.

205. It would be easy to give, for this Problem considered generally, some formulas nearly the same to those which I have given for the 1st Problem, art. 195. This would be again the same method; but there is enough on this matter, & one can not totally exhaust.

REMARK.

206. It had been difficult to employ for the solution of the preceding Problems another method than that of combinations, & changes of order. The Analysis applied to this Problem makes fall very naturally into paralogism: it is that which will appear for example as follows.

There lacks one point to Pierre & one point to Paul, Pierre plays with two quoits, Paul with one. One demands the lot of Pierre: in order to find by Analysis, here is how I reason. I suppose first that Paul has played his quoit: this done, Pierre plays the first of these quoits. Now it is clear that playing this first quoit he has $\frac{1}{2}$ in order to win, by making a better trial than that of Paul; & $\frac{1}{2}$ in order that his first trial not being so good as the one of Paul, he has no more hope than in this second trial. Now by assumption the forces being equals, his second quoit furnishes him $\frac{1}{2}$ expectation to win on the second trial, having not at all won on the first; & consequently the lot of Pierre is $\frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$; so that by following this reasoning there would be odds of 3 against 1 that Pierre would win rather than Paul.

It is clear & very true, that in the theory the case of equality of the bowls must be counted for nothing, & changes nothing of the preceding calculation; so that it can appear entirely exempt from paralogism. Here is nevertheless another reasoning very convincing which gives the ratio of the lots as 2 to 1.

I imagine two diamonds which represent the two quoits of Pierre, & one heart which represents the quoit of Paul, & I imagine that the Problem proposed is reduced to discover how much there is to wager that mixing the cards at random & drawing a card, that which will present itself first will be a diamond. Now the lot of the one who would wager to draw first a diamond, would be visibly $\frac{2}{3}$, & not $\frac{3}{4}$. Therefore, &.

This last calculation is founded on one very just & very simple idea. How therefore to accord this contrariety between two solutions which seem the one & the other to have the character of evidence? The last is the true. Here is a demonstration of it which leaves no doubt.

If one supposed that Pierre gave one of his two quoits to play to a third Player of like force as he, it is clear that this would carry no prejudice to Paul, & that in this case the lot of each of the three Players would be $\frac{1}{3}$; & consequently the one of Pierre $\frac{2}{3}$ when he will have the right of two Players who will have each a quoit; or, this which is the same thing, when he will have two quoits. Thus the secret paralogism of our 1st solution where we have found the lot of Pierre is $\frac{3}{4}$ is founded on that which in the analytic reasoning one supposes that when the first quoit of Pierre is not found so good as the one of Paul; Paul recommences to play his quoit against Pierre his own, instead that in the ordinary rule of the game, the one of Paul which has already won remaining; & by taking care, one perceives well that this is not indifferent, since it is to presume that the quoit of Paul must already have some sort of goodness, when it is found better than the first quoit of Pierre; & that thus he can take off without disadvantage his quoit in order to replay, since it is probable that he will not make a trial so good.