SOLUTIO GENERALIS PROBLEMATIS XV. *PROPOSITI À* D. DE MOIVRE*, IN TRACTATU* DE MENSURA SORTIS *INSERTO ACTIS PHILOSOPHICIS ANGLICANIS* NO 329 *PRO NUMERO QUOCUNQUE COLLUSORUM*

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Since the synthetic method, which D. *de Moivre* has used to discover the lot of each Gamester, may not be able to be altered in use then when there are more than three gamesters, on account of the law of the progression of the series which presents itself being observed with difficulty; I will reveal here by what manner Analysis in Problems of this kind, where the deposit is continuously increased, may be able to be employed: and I will give the analytic demonstration of three Theorems in the end, which I have discovered, & indeed long since before the book de Mensura Sortis of *Moivre* appeared, on the occasion of a triple question proposed to me by a friend concerning this game, which the French call *le Jeu de la Poule*, namely for the probability to be discovered of winning, likewise for the gain or the ruin of each Gamester, & for the duration of competition.

Theorem I.

If several Gamesters A, B, C, D, E &c. of whom the number is $n + 1$ & skills are equal, individually deposit 1, & compete with such conditions. 1° That of those the two A & B start the game. 2° That the defeated cedes his place to the third C, thus as that third C now competes with the victor, and who will have escaped victor from this contest will play with the fourth D and thus in succession. 3° That the one should obtain the entire deposit, who will have defeated all gamesters successively. I say the probabilities of winning of any two gamesters whosoever playing with themselves immediately in order of sequences to be in ratio $1 + 2^n$ to 2^n, and precisely the expectations of the players $A(B)$, C, D, E, &c. to be in Geometric progression.

Demonstration.

The expectations of winning of the one A or B may be put$=$ a, of the one $C = c$, of the one $D = d$, of the one $E = e$, &c. Further while it may be able to happen, that any gamester with the first turn entering into the game may find the opponent who either not yet, or once, or twice, or thrice, &c. now successively has been the victor, let the expectation of the player in the first case be called $= z$, in the second $= y$, in the third $= x$, in the fourth $= u$, in the fifth $= t$, &c. Likewise when any gamester is able to be defeated by the opponent who previously now has defeated either none, or one, or two, or three &c. gamesters, thus when exiting from the game the opponent must remain who either one, or two, or three, or four &c. has been the victor, let the expectation or probability of the defeated one be called in the first case $= h$, in the second $= k$, in the third $= l$, in the

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Date: *Philosophical Transactions* Vol. 29, pp. 133–144. *General Solution of Problem* XV. *proposed by* D. de Moivre*, in the tractate* de Mensura Sortis *inserted into Philosophical Transactions* No. 329 *for any number of gamesters*.

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fourth $= m$, &c. With all these things put there will be had the following nine series of equations designated No. 1, No. 2, No. 3, &c. up to No. 9. *Table* I. This reasoning is to be discovered shortly. It is discovered among the equations No. 1 *for example*

$$f = \frac{1}{8}t + \frac{1}{8}u + \frac{1}{4}x + \frac{1}{2}y.$$

For instance gamester F will compete either with gamester A, or B, or C, or D, or E: that the first or the second may happen, it is necessary that either$A$ or B become victor four times successively, of which event the probability is $\frac{2}{16}$ or $\frac{1}{8}$: That the third may happen it is necessary that C become victor thrice, the probability of which event is $\frac{1}{8}$ besides: that the fourth may happen it is necessary that D become victor twice successively, which has probability $\frac{1}{4}$; That the fifth may happen, it is necessary that E win once, of which event the probability is $\frac{1}{2}$; therefore the probability of winning of the player F is

$$= \frac{1}{8}t + \frac{1}{8}u + \frac{1}{4}x + \frac{1}{2}y.$$

Thus among the equations No. 2 is, *for example*

$$x = \frac{1}{2}l + \overline{\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \frac{1}{2^n}} \times h + \frac{1}{2^n} \times 1.$$

For the gamester who competes with an opponent who now has been the victor twice successively, is able to defeat either all gamesters, or some, or none. The probability of the prior event is $\frac{1}{2^n}$, of the second

$$\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \frac{1}{2^n},$$

& of the third $\frac{1}{2}$; if the first event should happen, the probability of winning results entire certitude or 1; if the second, the remaining gamester exits from the game who has defeated once; if the third, the remaining gamester exits from the game who has defeated thrice successively; and the lot of it in total is

$$\frac{1}{2} \times l + \overline{\frac{1}{4} + \frac{1}{8} + \frac{1}{16} \cdots \frac{1}{2^n}} \times h + \frac{1}{2^n} \times 1.$$

With a similar computation equations No. 3 are discovered. For the gamester who defeated by the opponent exits from the game, the remaining gamester *for example* victor of as many as one game, acquires the lot either of the one C, or of the one D, or of the one E, or of the one F, &c. just as the opponent by whom he has been defeated defeats either all gamesters except one, or all except two, or all except three, &c. whence it follows that

$$h = \frac{1}{2^{n-1}} \times c + \frac{1}{2^{n-2}} \times d + \frac{1}{2^{n-3}} \times e + \frac{1}{2^{n-4}} \times f + \&c.$$

Equations No. 4 are discovered by subtracting equations No. 2 from one another: & equations No. 5 by subtracting equations No. 3 from one another. Equations No. 6 are discovered by subtracting in equation No. 4 the values discovered through equations No. 5. Equations No. 7 are discovered by seeking the values of these z, y, x, u, &c. through equations No. 1. And with these values substituted into equations No. 4 equations No. 8 will be had, which comparisons with equations No. 6 give equations No. 9 from which it follows that

$$1 + 2^n.2^n :: a : c :: c : d :: d : e, \text{ \&c.} \quad \text{Q.E.D.}$$

Corollary

Hence easily the probabilities of winning are discovered of all Gamesters individually, which they have at the time before the game is undertaken, then into whatever state to which they are able to reach by pursuing the game. If there are, for example, three Gamesters A, B, C, there will be $n = 2$, & $1 + 2^n : 2^n :: 5.4 :: a.c$: that is, the probabilities of winning of the ones A, B, C, before A will have defeated B, or B will have defeated A, are themselves as the numbers 5, 5, 4, and precisely the probabilities themselves are $\frac{5}{14}$, $\frac{5}{14}$, $\frac{4}{14}$: in fact likewise all must be assumed to make 1 or entire certitude. After A will have defeated B, the probabilities of winning of the ones B,C, A, will be h, y or c, & (because A has equal expectation to victory, & to the lot to be obtained of the one B) $\frac{1+h}{2}$ respectively, *this is*, because through equation I No. 3.

$$h = \frac{1}{2^{n-1}} \times c = \frac{1}{2}c,$$

and

$$c = \frac{4}{14} = \frac{2}{7}$$

as discovered by the method, these probabilities will be $\frac{1}{7}$, $\frac{2}{7}$, $\frac{4}{7}$, as D. *de Moivre* has discovered Corollary I. Prop. 15. pg. 242.

If there are four gamesters A, B, C, D, there will be $n = 3$, & $1 + 2^n.2^n :: 9 : 8$, and precisely the probabilities of the gamesters at the beginning of the game will be as 9, 9, 8, $\frac{8\times 8}{9}$, or as 81, 81, 72, 64, *this is*, the ones a, a, c, d, will be $\frac{81}{298}$, $\frac{81}{298}$, $\frac{72}{298}$ & $\frac{64}{298}$. After A will defeat B, the probabilities of the ones B, D, C, A, will be $h\,d$, c, $\frac{1+3h}{4}$, but there is through equation 1. No. 3

$$\begin{aligned} h &= \frac{1}{2^{n-1}} \times c + \frac{1}{2^{n-2}} \times d = \frac{1}{4}c + \frac{1}{2}d, \\ c &= \frac{72}{298} = \frac{36}{149}, \\ \& \quad d &= \frac{64}{298} = \frac{32}{149}, \end{aligned}$$

as discovered by the method: therefore these probabilities will be $\frac{25}{149}$, $\frac{32}{149}$, $\frac{36}{149}$, $\frac{56}{149}$ respectively. After A will have defeated B & C, the probabilities of winning of the ones C, B, D, A, will be k, $\frac{c}{2}$, x, $\frac{1+h}{2}$, or (because through equation 2 No. 3

$$k = \frac{1}{2^{n-2}} \times d = \frac{1}{2}d,$$

& through equation 3 No. 7 $x = 2d - c$) $\frac{16}{149}$, $\frac{18}{149}$, $\frac{28}{149}$, $\frac{87}{149}$. And note that the integrity of the calculation may confirm from it, that the sum of these probabilities, *this is*, $\frac{1}{7} + \frac{2}{7} + \frac{4}{7}$ in the prior example, & $\frac{25}{149} + \frac{32}{149} + \frac{36}{149} + \frac{56}{149}$, and also $\frac{16}{149} + \frac{18}{149} + \frac{28}{149} + \frac{87}{149}$ in the last example, are individually $= 1$ or entire certitude.

Theorem II.

Whatever prior & above with this condition posed, that the loser always must be fined the sum p, which serves to increase the deposit; because he must cede the deposit thus by degrees increased to that one alone, who will have been the victor of all the gamesters successively; for with the probabilities of winning denoted as before by the lowercase letters a, c, d, e, &c. of the ones A (or B), C, D, E, &c.: indeed with the expectations by the very same uppercase letters A, C, D, E, &c. of the ones A (or B), C, D, E, &c., *this* *is*, with the proportions of the deposit which they expect individually: I say, always to be

$$C = \frac{\overline{A + ap} \times 2^n - ncp}{1 + 2^n}$$
$$D = \frac{\overline{C + cp} \times 2^n - ndp}{1 + 2^n}$$
$$E = \frac{\overline{D + dp} \times 2^n - nep}{1 + 2^n}, \text{ \&c.}$$

Demonstration.

Let the probability of the player winning be denoted as before by the lowercase letters z, y, x, u, t, &c together with the opponent, who now has defeated either none, or one, or two &c. gamesters successively; by the same actual uppercase letters Z, Y, X, U, T &c. the expectation of it, how much namely he has with these diverse cases, with the deposit being $n+1$, $n+1+p$, $n+1+2p$, $n+1+3p$, &c. respectively. Thus besides by the lowercase letters h, k, l, m, &c. let be denoted the probability of winning of the player defeated by the opponent, who before had defeated either none, or one, or two, &c. successive gamesters; as by the uppercase letters H, K, L, M, &c. the expectation of the same with these diverse cases, with the deposit being $n + 1$, $n + 1 + p$, $n + 1 + 2p$, $n + 1 + 3p$, &c. respectively. With these same posed by which calculations previously the following series of twelve equations in *Table* II designated No. 1, No. 2, No. 3, &c. will be discovered. Among the equations No. 1 for example is

$$E = \frac{U}{4} + \frac{X + xp}{4} + \frac{Y + 2yp}{2}.$$

For player E will play either with player A, or with player B, or C, or D. If he plays with A or B, the expectation of him will be U, because he plays with an opponent who now has defeated three opponents, with the deposit being $n + 1 + 3p$. If he plays with player C, the expectation of him will be $= X + xp$, for he plays with an opponent who now has defeated two gamesters, and much more if the deposit of him was $n + 1 + 2p$, the expectation was $= X$: indeed because with E playing the deposit is $= n + 1 + 3p$, on account of three gamesters defeated & the sum p fined, that portion of the fine p is added to the expectation X, how much the player E is able to expect: but this portion is (because the probability of him winning is x) $= xp$, therefore the total expectation of him then will be $= X + xp$. Thus if he wins with player D, the expectation of him will be $= Y + 2yp$: add to Y (which was the expectation of him with the deposit being $n+1+p$) the portion $2yp$, which is owed to him from the two fines $2p$, by which the deposit $n + 1 + 3p$ is greater than $n + 1 + p$. In a similar manner the equations No. 2.3.4 & 5 are had. But equations No. 6 are had by the first equation No. 2 *Table* 1 being substituted into equations No. 4. And equations No. 7 are had by the first equation No. 3 *Table* I being substituted into equations No. 5 by which thence substituted into equations No. 6 are had equations No. 8. Equations No. 9 are discovered by seeking the values of the ones Z, Y, X, U, &c. through equations No. 1 *Tables* I & II or No. 2 *Table* II & No. 7 *Table* I. And equations No. 10 are had by these values substituted into equations No. 4. Which compared with equations No. 8 (in which a should be substituted for z, by equation 1 *Table* I) gives equations No. 11. And these equations No. 11 compared with equations No. 9 *Table* I give equations No. 12, which constitute the Theorem, what had to be demonstrated.

Corollary.

Hence likewise the lots or expectations of each Gamester are discovered easily, and of themselves to such degree gain or loss. Let there be *for example* three gamesters A, B, C: there will be

$$\begin{aligned} C =& \frac{\overline{A+ap} \times 2^n - ncp}{1+2^n} \\ &= \text{(on account } n=2) \ \frac{4A+4ap-2cp}{5} \\ &= \text{(on account } a = \frac{5}{14} \ \& \ c = \frac{2}{7} \text{ by cor. Theorem I.)} \ \frac{4A+\frac{6}{7}p}{5}. \end{aligned}$$

Whence with the expectations of all three assumed simultaneously, *that is*, $A+A+C$ must equal it because from the beginning it has been deposited, that is 3, it will be

$$2A + \frac{4A+\frac{6}{7}p}{5} = \frac{14A+\frac{6}{7}p}{5} = 3,$$

& $14A = 15 - \frac{6}{7}p$, & $A = \frac{15}{14} - \frac{3}{49}p =$ to the expectation of player A or B: hence C the expectation of the third player $C = \frac{4A+\frac{6}{7}p}{5} = \frac{6}{7} + \frac{6}{49}p$. By which expectations if 1 is subtracted, it what from the beginning they individually have deposited, will remain there $\frac{1}{14} - \frac{3}{49}p$, here $\frac{6}{49}p - \frac{1}{7}$; just as D. *de Moivre* discovers. *Example* 2. Let there be 4 gamesters, A, B, C, D, there will be

$$\begin{aligned} C =& \frac{\overline{A+ap} \times 2^n - ncp}{1+2^n} \\ &= \text{(on account } n=3) \ \frac{8A+8ap-3cp}{9} \\ &= \text{(on account } a = \frac{81}{298} \ \& \ c = \frac{36}{149} \text{ by cor. Theorem I.)} \ \frac{8A+\frac{216}{149}p}{9}; \end{aligned}$$

likewise

$$\begin{aligned} D =& \frac{\overline{C+cp} \times 2^n - ndp}{1+2^n} \\ =& \frac{8C+8cp-3dp}{9} \\ &= \text{(on account } d = \frac{32}{149} \text{ by the same cor.)} \ \frac{8C+\frac{192}{149}p}{9}; \\ =& \frac{64A+\frac{3456}{149}p}{81} : \end{aligned}$$

whence will be had the equation

$$2A+C+D = 2A + \frac{8A+\frac{216}{149}p}{9} + \frac{64A+\frac{3456}{149}p}{9} = \frac{298A+\frac{5400}{149}p}{81} = 4,$$

or $149A + \frac{2700}{149}p = 162$, & $A = \frac{162}{149} - \frac{2700}{22201}p$. Hence

$$C = \frac{8A+\frac{216}{149}p}{9} = \frac{144}{149} + \frac{1176}{22201}p,$$

&

$$D = \frac{64A+\frac{3456}{149}p}{81} = \frac{128}{149} + \frac{4224}{22201}p.$$

But with unity 1 subtracted, what from the beginning of the game they individually have deposited, there will remain $\frac{13}{149} - \frac{2700}{22201}p$ for player A or B, $\frac{1176}{22201}p - \frac{5}{149}$ for C, & $\frac{4224}{22201}p - \frac{21}{149}$ for D; what individually they will require of profit or loss, just as the positive part surpasses in power of the negative, or contrary. By the same reason besides the lots will be had which they acquire into whatever state to which they are able to arrive by pursuing the game.

Theorem III.

With posed what before, if spectators Q, R, S, T, U, &c. may be present of whom the number is n units less than the number of players, and of whom the first Q asserts the contest to be about to be ended after $n+p$ completed games, R after $n+p-1$, S after $n+p-2$, T after $n+p-3$, U after $n+p-4$, &c. precisely, not before; and let q, r, s, t, u, &c. be the lots of Q, R, S, T, U, &c. themselves. I say $q = \frac{1}{2}r + \frac{1}{4}s + \frac{1}{8}t + \frac{1}{16}u$+&c.

Demonstration.

Let that gamester be called A, who is supposed to win after $n + p$ games: here he must enter into the game after p completed games, & besides play against an opponent, who now defeated either one, or two, or three &c. gamesters successively. Now when, in order that the first case may happen, & in order that gamester A may defeat all his fellow gamblers except one, *that is*, $n - 1$ gamesters successively, equally the probability is how much that the opponent of him must defeat $n - 1$ gamesters, *that is*, (because now he has been victor of one gamester) that the contest must be finished after $n+p-1$ completed games: and the probability of this event will be $= r$: the probability that the gamester A thus far defeated the opponent, *that is*, the contest must finish after $n + p$ games will be $= \frac{1}{2}r$. Thus, as the second; the case must appear, & in order that A defeated all gamesters except two, equally probable is how much in order that the game must finish after $n + p - 2$ games, and more in order that then A defeated thus far two gamesters, *that is*, in order that the contest must finish after $n + p$ games, the probability will be $= \frac{1}{4}s$. In the same manner, as, in the third case arising, A defeated all gamesters, the probability is $= \frac{1}{8}t$; as in the fourth $= \frac{1}{16}u$, &c. Whereby that indifferently the game must finish after $n + p$ games, the probability is

$$\frac{1}{2}r + \frac{1}{4}s + \frac{1}{8}t + \frac{1}{16}u + \text{\&c.} = q. \text{ Q. E. D.}$$

Corollary I.

It is discovered easily from here what is the probability that the contest must finish within any given number of games. For the series of fractions starting with the fraction $\frac{1}{2^{n-1}}$, of which the denominators increase in continuous double proportion, but the numerator of each fraction is the sum of the numerators of so many fractions immediately preceding as many as there are units in $n - 1$, gives all successive probabilities, that the contest must finish with n, $n + 1$, $n + 2$, $n + 3$ &c. games completed: & consequently if so many terms of this series should be added as many as there are units in $p + 1$, the sum of themselves will express the probability that the contest must finish at least by $n + p$ completed games. *For example*. If there are 4 gamesters, and indeed $n = 3$, this series will have

$$\frac{1}{4}, \frac{1}{8}, \frac{2}{16}, \frac{3}{32}, \frac{5}{64}, \frac{8}{128}, \frac{13}{256}, \frac{21}{512} \text{ \&c.}$$

By reason of which if another may happen

$$\frac{1}{4}, \frac{3}{8}, \frac{8}{16}, \frac{19}{32}, \frac{43}{64}, \frac{94}{128}, \frac{201}{256} \text{ \&c.}$$

the terms of which are the sums of the terms of the preceding series, the same terms will denote what kind is the probability that the contest must finish at least in 3, 4, 5, 6, &c. games.

Corollary 2.

Any term of the first series (with the first term excepted) & the sum of all terms, *that is*, whatever term of the posterior series, is able to be expressed through the general formula in this manner. If $n+1$ is the number of gamesters, & p is the number of terms, the last term of the first series will be

$$\frac{1}{2^n} - \frac{p-n+1}{1\times 2^{2n}} + \frac{p-2n\times p-2n+3}{1\times 2\times 2^{2n}} - \frac{p-3n\times p-3n+1\times p-3n+5}{1\times 2\times 3\times 2^{4n}} + \frac{p-4n\times p-4n+1\times p-4n+2\times p-4n+7}{1\times 2\times 3\times 4\times 2^{5n}} - \text{\&c.}$$

And the sum of all terms or the last term of the posterior series

$$= \frac{p+1}{1\times 2^n} - \frac{p-n\times p-n+3}{1\times 2\times 2^{2n}} + \frac{p-2n\times p-2n+1\times p-2n+5}{1\times 2\times 3\times 2^{3n}} - \frac{p-3n\times p-3n+1\times p-3n+2\times p-3n+7}{1\times 2\times 3\times 4\times 2^{4n}} + \text{\&c.}$$

Table I.

Enter			Exit			No. 1
	Lot			Lot		
0	z		1	h		$a = z$
1	y		2	k		$c = y$
2	x		3	l		$d = \frac{1}{2}x + \frac{1}{2}y$
3	u		5	m		$e = \frac{1}{4}u + \frac{1}{4}x + \frac{1}{2}y$
4	t					$f = \frac{1}{8}t + \frac{1}{8}u + \frac{1}{4}x + \frac{1}{2}y$

No. 2

$$z = \frac{1}{2} + \overline{\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \frac{1}{2^n}} \times h + \frac{1}{2^n} \times 1$$

$$y = \frac{1}{2}k + \overline{\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \frac{1}{2^n}} \times h + \frac{1}{2^n} \times 1$$

$$x = \frac{1}{2}l + \overline{\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \frac{1}{2^n}} \times h + \frac{1}{2^n} \times 1$$

$$u = \frac{1}{2}m + \overline{\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \frac{1}{2^n}} \times h + \frac{1}{2^n} \times 1$$

No. 3

$$h = \frac{1}{2^{n-1}} \times c + \frac{1}{2^{n-2}} \times d + \frac{1}{2^{n-3}} \times e + \frac{1}{2^{n-4}} \times f + \cdots$$

$$k = \frac{1}{2^{n-2}} \times d + \frac{1}{2^{n-3}} \times e + \frac{1}{2^{n-4}} \times f + \cdots$$

$$l = \frac{1}{2^{n-3}} \times e + \frac{1}{2^{n-4}} \times f + \cdots$$

$$m = \frac{1}{2^{n-4}} \times f + \cdots$$

No. 4	No. 6	No. 8
$z - y = \frac{1}{2}h - \frac{1}{2}k$	$= \frac{1}{2^n} \times c$	$= a - c$
$y - x = \frac{1}{2}k - \frac{1}{2}l$	$= \frac{1}{2^{n-1}} \times d$	$= 2c - 2d$
$x - u = \frac{1}{2}l - \frac{1}{2}m$	$= \frac{1}{2^{n-2}} \times e$	$= 4d - 4e$

No. 5	No. 7	No. 9
$h - k = \frac{1}{2^{n-1}} \times c$	$z = a$	$c = a \times \frac{2^n}{1+2^n}$
$k - l = \frac{1}{2^{n-2}} \times d$	$y = c$	$d = c \times \frac{2^n}{1+2^n}$
$l - m = \frac{1}{2^{n-3}} \times e$	$x = 2d - y = 2d - c$	$e = d \times \frac{2^n}{1+2^n}$
	$u = 4e - x - 2y = 4e - 2d - e$	

Corollary 3.

Before anyone is able to undertake by himself a game it must begin, that the sum $n+1$ of which the gamesters contend, & the fines all pay out, if $n + 1 + 2^n - 1 \times p$ must be given at the beginning to one another in hands.

I leave the demonstration of the preceding two corollaries to be investigated by the curious.

Table II.

Enter Deposit		Exit Lot	Deposit	No. 1	Lot	Deposit	
$n+1$	0	Z	$n+1+p$	1	H	$n+1+$	$A=Z$
$n+1+p$	1	Y	$n+1+2p$	2	K	$n+1+p$	$C=Y$
$n+1+2p$	2	X	$n+1+3p$	3	L	$n+1+2p$	$D=\frac{1}{2}X+\frac{1}{2}\times\overline{Y+yp}$
$n+1+3p$	3	V	$n+1+4p$	4	M	$n+1+3p$	$E=\frac{1}{4}V+\frac{1}{4}\times\overline{X+xp}+\frac{1}{2}\times\overline{Y+2yp}$
$n+1+4p$	4	T				$n+1+4p$	$F=\frac{1}{8}T+\frac{1}{8}\times\overline{V+vp}+\frac{1}{4}\times\overline{X+2xp}+\frac{1}{2}\times\overline{Y+3yp}$

No. 2

$$Z=\frac{1}{2}\overline{H-p}+\frac{1}{4}\times\overline{H-p+hp}+\frac{1}{8}\times\overline{H-p+2hp}+\frac{1}{16}\times\overline{H-p+3hp}+\cdots\frac{1}{2^n}\times\overline{H-p+nhp-hp}+\frac{1}{2^n}\times\overline{np+n+1}$$

$$Y=\frac{1}{2}\overline{K-p}+\frac{1}{4}\times\overline{H-p+2hp}+\frac{1}{8}\times\overline{H-p+3hp}+\frac{1}{16}\times\overline{H-p+4hp}+\cdots\frac{1}{2^n}\times\overline{H-p+nhp}+\frac{1}{2^n}\times\overline{np+p+n+1}$$

$$X=\frac{1}{2}\overline{L-p}+\frac{1}{4}\times\overline{H-p+3hp}+\frac{1}{8}\times\overline{H-p+4hp}+\frac{1}{16}\times\overline{H-p+5hp}+\cdots\frac{1}{2^n}\times\overline{H-p+nhp+hp}+\frac{1}{2^n}\times\overline{np+2p+n+1}$$

$$V=\frac{1}{2}\overline{M-p}+\frac{1}{4}\times\overline{H-p+4hp}+\frac{1}{8}\times\overline{H-p+5hp}+\frac{1}{16}\times\overline{H-p+6hp}+\cdots\frac{1}{2^n}\times\overline{H-p+nhp+hp}+\frac{1}{2^n}\times\overline{np+3p+n+1}$$

No. 3

$$H=\frac{1}{2^{n-1}}\times\overline{C+ncp-cp}+\frac{1}{2^{n-2}}\times\overline{D+ndp-2dp}+\frac{1}{2^{n-3}}\times\overline{E+nep-3ep}+\frac{1}{2^{n-4}}\times\overline{F+nfp-4fp}+\cdots$$

$$K=\frac{1}{2^{n-2}}\times\overline{D+ndp-dp}+\frac{1}{2^{n-3}}\times\overline{E+nep-2ep}+\frac{1}{2^{n-4}}\times\overline{F+nfp-3fp}+\cdots$$

$$L=\frac{1}{2^{n-3}}\times\overline{E+nep-ep}+\frac{1}{2^{n-4}}\times\overline{F+nfp-2fp}+\cdots$$

$$M=\frac{1}{2^{n-4}}\times\overline{F+nfp-2fp}+\cdots$$

No. 4

$$Y - Z = \frac{1}{2}K - \frac{1}{2}H + \frac{1}{4}hp + \frac{1}{8}hp + \frac{1}{16}hp + \cdots \frac{1}{2^n} \times hp$$
$$X - Y = \frac{1}{2}L - \frac{1}{2}K + \frac{1}{4}hp + \frac{1}{8}hp + \frac{1}{16}hp + \cdots \frac{1}{2^n} \times hp$$
$$V - X = \frac{1}{2}M - \frac{1}{2}L + \frac{1}{4}hp + \frac{1}{8}hp + \frac{1}{16}hp + \cdots \frac{1}{2^n} \times hp$$

No. 6

$$= \frac{1}{2}K - \frac{1}{2}H + zp - \frac{1}{2}hp$$
$$= \frac{1}{2}L - \frac{1}{2}K + zp - \frac{1}{2}hp$$
$$= \frac{1}{2}M - \frac{1}{2}L + zp - \frac{1}{2}hp$$

No. 8

$$= -\frac{1}{2^{n-0}} \times C - \frac{ncp}{2^n} + zp$$
$$= -\frac{1}{2^{n-1}} \times D - \frac{ndp}{2^{n-1}} - \frac{cp}{2^n} + zp$$
$$= -\frac{1}{2^{n-2}} \times E - \frac{nep}{2^n} - \frac{dp}{2^{n-1}} - \frac{cp}{2^n} + zp$$

No. 5

$$K - H = -\frac{1}{2^{n-1}} \times \overline{C + ncp - cp} + \frac{1}{2^{n-2}} \times dp + \frac{1}{2^{n-3}} \times cp + \frac{1}{2^{n-4}} \times fp + \cdots$$
$$L - K = -\frac{1}{2^{n-2}} \times \overline{D + ndp - dp} + \frac{1}{2^{n-3}} \times cp + \frac{1}{2^{n-4}} \times fp + \cdots$$
$$M - L = -\frac{1}{2^{n-3}} \times \overline{E + nep - ep} + \frac{1}{2^{n-4}} \times fp + \cdots$$

No. 7

$$= \frac{1}{2^{n-1}} \times C - \frac{ncp}{2^{n-1}} + hp$$
$$= \frac{1}{2^{n-2}} \times D - \frac{ndp}{2^{n-2}} - \frac{cp}{2^{n-2}} + hp$$
$$= \frac{1}{2^{n-1}} \times E - \frac{nep}{2^{n-3}} - \frac{dp}{2^{n-2}} - \frac{cp}{2^{n-1}} + hp$$

No. 9

$$Z = A$$
$$Y = C$$
$$X = 2D - C - cp$$
$$V = 4E - 2D - C - 2dp - 2cp$$

No. 10

$$C - A \quad = Y - Z$$
$$2D - 2C - cp \quad = X - Y$$
$$4E - 4D - 2dp - cp \quad = V - X$$

No. 8

$$= -\frac{1}{2^n} \times C - \frac{ncp}{2^n} + zp$$
$$= -\frac{1}{2^{n-1}} \times D - \frac{ndp}{2^{n-1}} - \frac{cp}{2^n} + zp$$
$$= -\frac{1}{2^{n-2}} \times E - \frac{nep}{2^{n-2}} - \frac{dp}{2^{n-1}} - \frac{cp}{2^n} + zp$$

No. 11

$$C = \frac{A \times 2^n + ap \times 2^n - ncp}{1 + 2^n}$$
$$D = \frac{C \times 2^n + cp \times \overline{2^{n-1} - \tfrac{1}{2}} + ap \times 2^{n-1} - ndp}{1 + 2^n}$$
$$E = \frac{D \times 2^n + dp \times \overline{2^{n-1} - \tfrac{1}{2}} + cp \times \overline{2^{n-2} - \tfrac{1}{4}} + ap \times 2^{n-2} - ncp}{1 + 2^n}$$

No. 12

$$= \frac{\overline{A + ap} \times 2^n - nap}{1 + 2^n}$$
$$= \frac{\overline{C + cp} \times 2^n - ndp}{1 + 2^n}$$
$$= \frac{\overline{D + dp} \times 2^n - ncp}{1 + 2^n}$$