EXAMEN ET RESOLUTION DE QUELQUES QUESTIONS SUR LES JEUX

FRANÇOIS NICOLE
HISTOIRE DE L'ACADEMIE ROYALE DES SCIENCES 1730, P. 45–56.

We can consider every game, which amusement or the desire to increase our money has invented, under two kinds. The first kind contains the games where chance alone has a part, & which by their nature places the players into different conditions, so that one has advantage over the other, as in the games of bassette, of pharaon, & of three dice, &c. The second kind contains the games where the chance being equal for the players as in piquet, &c. the strengths or degrees of ability among the players are different. 11 February 1730

Among the diverse Problems which we can propose on each of these two kinds of games, there is what are common to them, the greatest probability of winning for one of the players, being able to come equally from the nature of the game which gives to him advantage, or of the superiority of ability.

The question which we examine here is of this kind, it has been made to me many times by the high stakes players: here it is.

Two players play a part in any game,[1] for example at piquet; one of the players has more probability of winning this part than he has of losing it: we demand, when these players agree to play a certain number of parts, if the superior player has always the same advantage over the other, or the same degree of probability of winning more of the parts than the other; or if this probability increases, we demand according to what law is this increase made.

PROBLEM.

Two players, of whom the strengths are between them as p & q, play at piquet a certain number of parts, we demand what probability there is that the strongest player win, that which the players call the queüe des paris[2]*, & what is his advantage. The one who loses, is the one who is scored the most times in the course of the games than one has agreed to play.*

In order to resolve this Problem, it is necessary to discover first what is the advantage of this player; when one plays only two parts, next when one plays four of them, then six, eight, ten, & finally the number of which one is agreed. Because it is clear that his lot, when one plays twelve of them, for example, must result from the examination of the different states in which this game of cards can be found in every course of these twelve parts, & that any of these states correspond to the situation where the two players could be, if they played only to two parts, or to four, six, eight & ten.

SOLUTION.

I call *Pierre* the first player, of whom the strength is expressed by p, & *Paul* the second player, of whom the strength is expressed by q; p is greater than q.

Date: October 4, 2009.
Translated by Richard J. Pulskamp, Department of Mathematics and Computer Science, Xavier University, Cincinnati, OH. .

[1] *Translator's note*: "une partie à un jeu quelconque," that is, the *partie* is a subgame.

[2] the line or extreme of the wagers.

Let be supposed that they play first to two parts, let the money that one wins be named a, when one wins the wager. This put:

If we name s the lot of Pierre that we seek, x his lot when he wins the first part, & y when he loses it, we will have these equations

$$s = \frac{\overset{1.0}{p \times x} + \overset{0.1}{q \times y}}{p+q}, \quad x = \frac{\overset{2.0}{p \times a} + \overset{1.1}{q \times 0}}{p+q} \quad \& \quad y = \frac{\overset{1.1}{p \times 0} + \overset{0.2}{q \times -a}}{p+q},$$

in which the numbers which are written above each member of these equations, serve to express that which each player has won of parts. Therefore

$$s = \frac{p \times ap + q \times -aq}{(p+q)^2} = \frac{app - aqq}{(p+q)^2},$$

which is the lot of Pierre, or his advantage, when one plays to two parts.

Let be put now that these players play to four parts.

If we name s the lot of Pierre which we seek, x his lot when he wins the first part, y when he loses it; z his lot, when having won the first part, he wins again the second; r his lot, when having won the first two parts, he loses the third, we will have these Equations,

$$s = \frac{\overset{1.0}{p \times x} + \overset{0.1}{q \times y}}{p+q}, \qquad x = \frac{p \times z + q \times \overset{1.1}{\frac{app-aqq}{(p+q)^2}}}{p+q},$$
$$z = \frac{\overset{3.0}{p \times a} + \overset{2.1}{q \times r}}{p+q}, \qquad r = \frac{\overset{1.1}{p \times a} + \overset{2.2}{q \times 0}}{p+q} = \frac{ap}{p+q}.$$

Therefore,

$$z = \frac{ap}{p+q} + \frac{apq}{(p+q)^2} = \frac{app + 2apq}{(p+q)^2}$$

and

$$x = \frac{ap^3 + 2appq + appq - aq^3}{(p+q)^3} = \frac{ap^3 + 3appq - aq^3}{(p+q)^3}.$$

In order to determine y, let next the lot of Pierre be named t, when having lost the first part, he loses again the second, & u his lot, when having lost the first two parts, he wins the third, we will have these new Equations,

$$y = \frac{p \times \overset{1.1}{\frac{app-aqq}{(p+q)^2}} + \overset{0.2}{q \times t}}{p+q}, \quad t = \frac{\overset{1.2}{p \times u} + \overset{0.3}{q \times -a}}{p+q}, \quad u = \frac{\overset{2.2}{p \times 0} + \overset{1.3}{q \times -a}}{p+q} = -\frac{qa}{p+q}.$$

Therefore,

$$t = -\frac{apq}{(p+q)^2} - \frac{qa}{p+q} = -\frac{2apq - aqq}{(p+q)^2}$$

&

$$y = \frac{ap^3 - apqq - 2apqq - aq^3}{(p+q)^3} = \frac{ap^3 - 3apqq - aq^3}{(p+q)^3},$$

& by substituting into the first Equation for x & y their values, there comes

$$s = \frac{ap^4 + 3ap^3q - apq^3 + ap^3q - 3apq^3 - aq^4}{(p+q)^4} = \frac{ap^4 + 4ap^3q - 4apq^3 - aq^4}{(p+q)^4}$$

which is the sought lot of Pierre, or his advantage, when one plays to four parts.

If we suppose now that one plays to six parts, by employing as many unknowns as we have need, we will have all the following equations,

$$s = \frac{\overset{1.0}{p \times x} + \overset{0.1}{q \times y}}{p+q}, \quad x = \frac{\overset{2.0}{p \times z} + q \times \overset{1.1}{\frac{ap^4+4ap^3q-4apq^3-aq^4}{(p+q)^4}}}{p+q}, \quad z = \frac{\overset{3.0}{p \times r} + \overset{2.1}{q \times t}}{p+q},$$

$$r = \frac{\overset{4.0}{p \times a} + \overset{3.1}{q \times u}}{p+q}, \quad u = \frac{\overset{4.1}{p \times a} + \overset{3.2}{q \times m}}{p+q}, \quad m = \frac{\overset{4.2}{p \times a} + \overset{3.3}{q \times 0}}{p+q} = \frac{ap}{p+q}.$$

Therefore

$$u = \frac{ap}{p+q} + \frac{apq}{(p+q)^2} = \frac{app+2apq}{(p+q)^2},$$

$$r = \frac{ap}{p+q} + \frac{appq+2apqq}{(p+q)^3} = \frac{ap^3+3appq+3apqq}{(p+q)^3}.$$

In order to determine t, we have

$$t = \frac{p \times \overset{3.1}{\frac{app+2apq}{(p+q)^2}} + q \times \overset{2.2}{\frac{app-aqq}{(p+q)^2}}}{p+q} = \frac{ap^3+3appq-aq^3}{(p+q)^3}.$$

Therefore,

$$z = \frac{ap^4+4ap^3q+6appqq-aq^4}{(p+q)^4}.$$

Whence finally we have

$$x = \frac{ap^5+5ap^4q+10ap^3qq-5apq^4-aq^5}{(p+q)^5}.$$

In order to determine y, we have these other Equations

$$y = \frac{p \times \overset{1.1}{\frac{ap^4+4ap^3q-4apq^3-aq^4}{(p+q)^4}} + \overset{0.2}{q \times k}}{p+q},$$

$$k = \frac{\overset{1.2}{p \times f} + \overset{0.3}{q \times h}}{p+q}, \qquad f = \frac{p \times \overset{2.2}{\frac{app-aqq}{(p+q)^2}} + \overset{1.3}{q \times e}}{p+q},$$

$$e = \frac{\overset{2.3}{p \times d} + \overset{1.4}{q \times -a}}{p+q}, \qquad d = \frac{\overset{3.3}{p \times 0} + \overset{2.4}{q \times -a}}{p+q} = -\frac{qa}{p+q}.$$

Therefore

$$e = -\frac{pqa}{(p+q)^2} - \frac{qa}{p+q} = -\frac{2apq-aqq}{(p+q)^2},$$

therefore

$$f = \frac{ap^3-3apqq-aq^3}{(p+q)^3};$$

we have also

$$h = \frac{p \times -\overset{1.1}{\frac{2apq-aqq}{(p+q)^2}} + \overset{0.4}{q \times -a}}{p+q} = -\frac{3appq-3apqq-aq^3}{(p+q)^3},$$

therefore

$$k = \frac{ap^4-6appqq-4apq^3-aq^4}{(p+q)^4},$$

& finally

$$y = \frac{ap^5 + 5ap^4q - 10appq^3 - 5apq^4 - aq^5}{(p+q)^5}.$$

And by substituting into the first equation, for x & y, the found values, it becomes

$$s = \frac{ap^6 + 6ap^5q + 15ap^4qq - 15appq^4 - 6apq^5 - aq^6}{(p+q)^6}$$

for the sought lot of Pierre, or his advantage, when one plays to six parts.

Let be supposed now that one plays to eight parts, & let A represent the lot of Pierre, when one plays to two parts, B when one plays to four parts, C when one plays to six parts.

If we use as many unknowns as we have need of them, we will have all the following equations,

$$s = \frac{p \times \overset{1.0}{x} + q \times \overset{0.1}{y}}{p+q}, \qquad x = \frac{p \times \overset{2.0}{z} + q \times \overset{1.1}{C}}{p+q}, \qquad z = \frac{p \times \overset{3.0}{u} + q \times \overset{2.1}{t}}{p+q},$$

$$u = \frac{p \times \overset{4.0}{a} + q \times \overset{3.1}{n}}{p+q}, \qquad r = \frac{p \times \overset{5.0}{a} + q \times \overset{4.1}{m}}{p+q}, \qquad m = \frac{p \times \overset{5.1}{a} + q \times \overset{4.2}{k}}{p+q},$$

$$k = \frac{p \times \overset{5.2}{a} + q \times \overset{4.3}{l}}{p+q}, \qquad l = \frac{p \times \overset{5.3}{a} + q \times \overset{4.4}{0}}{p+q} = \frac{ap}{p+q}.$$

Therefore

$$k = \frac{app + 2apq}{(p+q)^2}, \qquad m = \frac{ap^3 + 3appq + 3apq^2}{(p+q)^3}$$

&

$$r = \frac{ap^4 + 4ap^3q + 6appqq + 4apq^3}{(p+q)^4}.$$

We will find also

$$n = \frac{p \times \overset{4.1}{\overline{\frac{ap^3+3appq+3apqq}{(p+q)^3}}} + q \times \overset{3.2}{h}}{p+q},$$

$$h = \frac{p \times \overset{4.2}{\overline{\frac{app+2apq}{(p+q)^2}}} + q \times \overset{3.3}{A}}{p+q} = \frac{ap^3 + 3appq - aq^3}{(p+q)^3}.$$

Therefore

$$n = \frac{ap^4 + 4ap^3q + 6appqq + aq^4}{(p+q)^4};$$

& by substituting for h & n the values which we just found, it becomes[3]

$$u = \frac{ap^5 + 5ap^4q + 10ap^3qq + 10appq^3 - aq^5}{(p+q)^5}.$$

In order to to find the value of t, we have

$$t = \frac{p \times \overset{3.1}{\overline{\frac{ap^4+4ap^3q+6appqq-aq^4}{(p+q)^4}}} + q \times \overset{2.2}{B}}{p+q} = \frac{ap^5 + 5ap^4q + 10ap^3q^2 - 5apq^4 - aq^5}{(p+q)^5}.$$

[3]Nicole, in error, calls this n.

Therefore
$$z=\frac{ap^6+6ap^5q+15ap^4qq+20ap^3q^3-6apq^5-aq^6}{(p+q)^6},$$
& finally[4]
$$x=\frac{ap^7+7ap^6q+21ap^5qq+35ap^4q^3-21appq^5-7apq^6-aq^7}{(p+q)^7}.$$

In order to determine y, we have all these equations

$$y=\frac{p\overset{1.1}{\times C}+q\overset{0.2}{\times g}}{p+q},\qquad g=\frac{p\overset{1.2}{\times f}+q\overset{0.3}{\times e}}{p+q},\qquad f=\frac{p\overset{2.2}{\times B}+q\overset{1.3}{\times d}}{p+q},$$
$$d=\frac{p\overset{2.3}{\times c}+q\overset{1.4}{\times b}}{p+q},\qquad c=\frac{p\overset{3.3}{\times A}+q\overset{2.4}{\times Y}}{p+q},\qquad Y=\frac{p\overset{3.4}{\times X}+q\overset{2.5}{\times -a}}{p+q},$$
$$X=\frac{p\overset{4.4}{\times 0}+q\overset{3.5}{\times -a}}{p+q}=-\frac{aq}{p+q}.$$
Therefore
$$Y=-\frac{2apq-aqq}{(p+q)^2},$$
&
$$c=\frac{ap^3-3apqq-aq^3}{(p+q)^2}.$$
In order to find the value of b we have
$$b=\frac{p\overset{2.4}{\times Z}+q\overset{1.5}{\times -a}}{p+q},$$
$$Z=\frac{p\overset{3.4}{\times -\frac{aq}{p+q}}+q\overset{2.5}{\times -a}}{p+q}=-\frac{2apq-aqq}{(p+q)^2},$$
therefore
$$b=-\frac{3appq-3apq^2-aq^3}{(p+q)^3};$$
thus by substituting the values of b & c, there arrives
$$d=\frac{ap^4-6appqq-4apq^3-aq^4}{(p+q)^4},$$
therefore
$$f=\frac{ap^5+5ap^4q-10appq^3-5apq^4-aq^5}{(p+q)^5}.$$
In order to find the value of e, we have
$$e=\frac{p\times\overset{1.1}{\overline{\frac{ap^4-6appqq-4apq^3-aq^4}{(p+q)^4}}}+q\overset{0.4}{\times V}}{p+q},$$
$$V=\frac{p\times-\overset{1.4}{\overline{\frac{3appq-3apqq-aq^3}{(p+q)^5}}}+q\overset{0.5}{\times -a}}{p+q}=-\frac{4ap^3q-6appqq-4apq^3-aq^4}{(p+q)^4},$$

[4]Nicole, in error, calls this z.

therefore

$$e = \frac{ap^5 - 10ap^3qq - 10appq^3 - 5apq^4 - aq^5}{(p+q)^5}.$$

Now if we substitute for f & e the found values, we will have

$$g = \frac{ap^6 + 6ap^5q - 20ap^3q^3 - 15appq^4 - 6apq^5 - aq^6}{(p+q)^6}.$$

Therefore finally

$$y = \frac{ap^7 - 7ap^6q + 21ap^5qq - 35ap^3q^4 - 21ap^2q^5 - 7apq^6 - aq^7}{(p+q)^7};$$

and by substituting into the first equation, for x & y the found values, there arrives

$$s = \frac{ap^8 + 8ap^7q + 28ap^6q^2 + 56ap^5q^3 - 56ap^3q^6 - 28appq^6 - 8apq^7 - aq^8}{(p+q)^8}$$

which is the sought lot of Pierre, or his advantage, when one plays to eight parts.

We could, by the same way, determine the lot of Pierre, when one plays to ten parts, & next when one plays a greater number of them: but the number of equations, which it would be necessary run through in order to resolve these other cases, becoming quite considerable, it is simpler, in order to resolve them, to examine the magnitudes which have been found for the preceding cases, to compare them among themselves, & to discover by this comparison the law according to which they grow. The magnitudes, which have been found, are

$$\frac{app - aqq}{(p+q)^2}, \qquad \text{for 2 parts.}$$

$$\frac{ap^4 + 4ap^3q - 4apq^3 - aq^4}{(p+q)^4}, \qquad \text{for 4 parts.}$$

$$\frac{ap^6 + 6ap^5q + 15ap^4qq - 15appq^4 - 6apq^5 - aq^6}{(p+q)^6}, \qquad \text{for 6 parts.}$$

$$\frac{ap^8 + 8ap^7q + 28ap^6qq + 56ap^5q^3 - 56ap^3q^6 - 28appq^6 - 8apq^7 - aq^8}{(p+q)^8}, \qquad \text{for 8 parts.}$$

which are reduced, by dividing the numerators & denominators by $p+q$,

to $$\frac{ap-aq}{p+q}, \qquad \text{for 2 parts.}$$

to $$\frac{ap^3+3appq-3apqq-aq^3}{(p+q)^3}, \qquad \text{for 4 parts.}$$

to $$\frac{ap^5+5ap^4q+10ap^3qq-10appq^3-5apq^4-aq^5}{(p+q)^5}, \qquad \text{for 6 parts.}$$

to $$\frac{ap^7+7ap^6q+21ap^5qq+35ap^4q^3-35ap^3q^4-21appq^5-7apq^6-aq^7}{(p+q)^7}, \qquad \text{for 8 parts.}$$

to $$\frac{\begin{array}{r}ap^9+9ap^8q+36ap^7qq+84ap^6q^3+126ap^5q^4-126ap^4q^5\\ -84ap^3q^6-36appq^7-9apq^8-aq^9\end{array}}{(p+q)^9}, \qquad \text{for 10 parts.}$$

to $$a\times\left\{\frac{\begin{array}{r}p^{11}+11p^{10}q+55p^9q^2+165p^8q^3+330p^7q^4+462p^6q^5-462p^5q^6\\ -330p^4q^7-165p^3q^8-55p^2q^9-11pq^{10}-q^{11}\end{array}}{(p+q)^{11}}\right., \qquad \text{for 12 parts.}$$

We will have for 24 parts, or twelve Kings,

$$a\times\frac{\left\{\begin{array}{r}p^{23}+23p^{22}q+253p^{21}q^2+1771p^{20}q^3+8855p^{19}q^4+33649p^{18}q^5\\ +100947p^{17}q^6+245157p^{16}q^7+490314p^{15}q^8+817190p^{14}q^9\\ +1144066p^{13}q^{10}+1352078p^{12}q^{11}-1352078p^{11}q^{12}-1144066p^{10}q^{13}\\ -817190p^9q^{14}-490314p^8q^{15}-245157p^7q^{16}-100947p^6q^{17}\\ -33649p^5q^{18}-8855p^4q^{19}-1771p^3q^{20}-253p^2q^{21}-13pq^{23}-q^{23}\end{array}\right.}{(p+q)^{23}}$$

If $p=5$ & $q=4$, we will have

$$a\times\frac{5-4}{9}=\frac{1}{9}a \qquad \text{for 2 parts.}$$
$$a\times\frac{425-304}{729}=\frac{121a}{729} \qquad \text{for 4 parts.}$$
$$a\times\frac{35625-23424}{59049}=\frac{12201a}{59049} \qquad \text{for 6 parts.}$$
$$a\times\frac{2965625-1817344}{4782969}=\frac{1148281a}{4782969} \qquad \text{for 8 parts.}$$
$$a\times\frac{242359625-141604864}{383964489}=\frac{100754761a}{383964489} \qquad \text{for 10 parts.}$$
$$a\times=\frac{203142265625-11066793984}{31381059609}=\frac{9247471641a}{31381059609} \qquad \text{for 12 parts.}$$
And $$a\times\frac{617677182163500976625}{8862938119652501095929}, \text{ or about } a\times\frac{193}{277} \qquad \text{for 24 parts.}$$

This last quantity is between $\frac{2}{3}a$ & $\frac{3}{4}a$. Thus under the assumption, that the strengths or abilities of the players are as 5 to 4, the advantage that the strongest player has over the weakest is only the ninth part of that which is in the game, when they play to two parts, & this advantage becomes a little more than two thirds of that which is in the game, when they play to twenty-four parts. When therefore under this assumption two players play at piquet, & put into the contest each nine louis for that which we call the queue des paris (line of wager), the weakest player presents to the other 6 louis 13 liv. 0 f. 2 den. of the nine louis which he has put into the game.

REMARK I.

The quantities which have been found in the cases that we just examined, & which express the advantage of the strongest player: these quantities, I say, being composed of positive terms & of negative terms, it is clear that the sum of all the positives will express the lot of the strongest player, or his right in the game, & that the sum of all the negatives will express the lot of the weakest, or the right which he has in this contest: because the advantage is nothing other than the excess of the lot of the one over the lot of the other.

Thus, for two parts, the lot of the one will be $\frac{p}{p+q} \times a$. And the lot of the other will be $\frac{q}{p+q} \times a$.

For four parts

$$\frac{p^3 + 3ppq}{(p+q)^3} \times a \quad \text{and} \quad \frac{3pqq + q^3}{(p+q)^3} \times a.$$

For six parts

$$\frac{p^5 + 5p^4q + 10p^3qq}{(p+q)^5} \times a \qquad \text{And } \frac{10ppq^5 + 5pq^4 + q^5}{(p+q)^5} \times a.$$

For eight parts

$$\frac{p^7 + 7p^6q + 21p^5qq + 35p^4q^3}{(p+q)^7} \times a. \text{ And } \frac{35p^3q^4 + 21ppq^5 + 7pq^6 + q^7}{(p+q)^7} \times a.$$

For ten parts

$$\frac{p^9+9p^8q+36p^7qq+84p^6q^3+126p^5q^4}{(p+q)^9} \times a$$

$$\text{And } \frac{126p^4q^5+84p^3q^6+36ppq^7+9pq^8+q^9}{(p+q)^9} \times a$$

Whence one sees that in order to have the lot of each of the two players, when they play any number of parts, it is necessary to raise the binomial $p + q$ to a power of which the exponent is less by one unit than the number of parts that one must play, to divide into two parts this binomial thus raised, of which the first will be composed of all the first terms until the middle, & the second, of all the last terms taken, from the middle. Each of these parts being the numerator of a fraction, of which the denominator is the entire power, will express the lot of each of the players, & the excess of the one of these fractions on the other will express the advantage of the strongest player.

REMARK II.

If one had sought by one similar way to that which we have followed here, the lot of the players & the advantage of the one on the other, when they play in an odd number of parts, one will have found the same formulas as we have found, by supposing this number of parts expressed by the even number which follows it, next that the lot is the same, be it that one plays in one or two parts: it is yet the same, be it that one plays in three or four parts, five or six parts, & thus of the others.

This can make trouble at first sight: because it is clear that the strongest player has so much more advantage as one plays in a greater number of parts: thus by this consideration he must have more advantage, when one plays to six parts, than when one plays to five parts. But this advantage is diminished in the case of six parts, in that this player, in order to win, must win two parts more than the other; because in this case, in order to win, it is necessary that he take four parts, & the other two; instead as in the case of five parts, it suffices that he take one part more than the other, that is to say, three parts, & the other two: thus, by this second consideration, the advantage of the strongest player must be diminished, because it is evident that it is more difficult to win two parts more than the other, than it is to win only one more of them. This reflection suffices to show the possibility of what the calculation gives; because the same reasoning will have place for all even number parts compared to the odd number which precedes it.

COROLLARY I.

If we suppose $p = q$, & that we substitute into the table which expresses the advantage of the strongest player, for q, its value p, we will see that this advantage becomes null in all cases, that is to say, whatever be the number of parts that one plays. And if we substitute p in the place of q, in the table which expresses the lot of the two players, we will find $\frac{1}{2}$ for the lot of each player, whatever be the number of parts that one plays, & this is also that which must happen.

COROLLARY II.

If before the end of the parts that one has agreed to play, one was obligated to quit the game, & if one wished to discover in what manner it is necessary to share the money of the game relatively to the state where the part is, when one ceases to play: we will find in what manner it is necessary to make this sharing, & what is the advantage or the disadvantage of the players, by examining among all the equations that one has been obligated to examine, which is the one which contains the proposed case, & this equation will give that which one seeks.

If one demands, for example, what is the advantage of Pierre, who is the strongest player, when the state of the part is such, that playing to eight parts, this player has four of them, & the other two, the equation of this case has been found

$$k = \frac{app + 2apq}{(p+q)^2}$$

for the advantage of Pierre, which under the assumption $p = 5$ & $q = 4$, gives $\frac{65}{81}a$ for this advantage; whence it follows that that which belongs to one of the players is $\frac{73}{81}a$, & that which belongs to the other is $\frac{8}{81}a$, that is to say, that it is necessary that Paul give to Pierre $\frac{65}{81}a$.

If the state of the contest is such, that Pierre has two points, & Paul four, when one plays to eight points, the equation of this case is $Y = -\frac{2apq - aqq}{(p+q)^2}$, which is the advantage of Pierre; but as this quantity is negative, it expresses that which Pierre must pay to Paul, or the advantage of Paul, which under the assumption of $p = 5$ & $q = 4$, is $-\frac{56}{81}a$, that is to say, that Pierre must pay to Paul $\frac{56}{81}a$, & the lots will be as $\frac{25}{162}$ to $\frac{137}{162}$. There will be thus of the others of them as we would wish to imagine.