Sur la probabilité des hypothèses d'après les événements*

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The method for seeking the probability of future events from past events has been proposed by Bayes and Price in the *Philosophical Transactions* for 1763 and 1764. Laplace has served himself first in order to treat some very important and very varied questions which correspond to it. But it does not appear to us that the illustrious geometer has demonstrated in full rigor, the principle which serves as base to the researches on this object, and which is summarized by the fundamental question that is here:

One awaits an event, which nonetheless may not take place, its arrival is explainable by n different hypotheses h_1, h_2, h_3,… h_n. These hypotheses are the only possible and they are mutually exclusive, that is that it would be contradictory to admit two or a greater number of them simultaneously. A certain chance different from zero is attached to the existence of each of them, and also each hypothesis gives some chances to the arrival of the event, but among the numbers of these last, numbers proper to each hypothesis, are able to be found which are equal to zero. The awaited event arrives, then one of the hypotheses h_1, h_2, h_3,… h_n has taken place, to find the probability, that it is a hypothesis indicated at will h_i.

Before exposing that which does not appear to us entirely rigorous in the analysis of Laplace, we are going to proceed to the solution of the question that we just proposed.

We designate by S the number of chances which exist before the arrival of the event and of which each brings forth one of the hypotheses h_1, h_2, h_3,… h_n; we will suppose these chances equally possible, that which is permitted, because one is always able to equalize the possibilities of them by subdivision. We suppose that of S chances s_1 lead to the hypothesis h_1, s_2 to the hypothesis h_2 and so forth to s_n chances which lead to the hypothesis h_n. None of the numbers s_1, s_2, s_3, … s_n is able to be zero, and none of the chances are able to belong to two or many hypotheses at the same time, otherwise those here would not be excluded. We have evidently

$$S = s_1 + s_2 + s_3 \cdots + s_n.$$

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In the s_1 chances favorable to the hypothesis h_1, we distinguish those which are at the same time favorable to the event from those which are contrary to it. Let f_1 be the number of the first, the one of the others will be clearly $s_1 - f_1$. Each of f_1 chances leading to the hypothesis h_1 will bring forth also the event, and $s_1 - f_1$ all other chances of them leading to the hypothesis h_1 exclude the event. We designate likewise by f_2, $f_3, \ldots f_n$ the chances included respectively in $s_2, s_3, \ldots s_n$ and favorable to the event; so that for example each of f_n chances included in s_n lead to the hypothesis h_n and at the same time makes the event arrive, but $s_n - f_n$ other chances each make only the hypothesis h_n arrive, not the event. We suppose:

$$f_1 + f_2 + f_3 + \cdots + f_n = F.$$

It is clear that the ratios

$$\frac{s_1}{S}, \quad \frac{s_2}{S}, \quad \frac{s_3}{S}, \ldots \frac{s_n}{S}$$

will represent respectively the probabilities of the hypotheses before the arrival of the event. It is clear also that

$$\frac{F}{S}$$

measures the probability of the event a priori, or by virtue of F chances which are favorable to it out of the total number of S equally possible chances. As for the ratio

$$\frac{f_1}{s_1}$$

it will represent the probability of the event by virtue of the hypothesis h_1 that is by considering this hypothesis as certain; and if one multiplies the preceding probability by the probability

$$\frac{s_1}{S}$$

of the hypothesis h_1 itself, one will have in the product

$$\frac{s_1}{S} \cdot \frac{f_1}{s_1}$$

the measure of the probability of the concurrence of the hypothesis h_1 and of the event; that is the a priori probability that the event will take place by virtue of the hypothesis h_1. Likewise the ratios

$$\frac{f_2}{s_2}, \quad \frac{f_3}{s_3}, \ldots \frac{f_n}{s_n}$$

and

$$\frac{s_2}{S} \cdot \frac{f_2}{s_2}, \quad \frac{s_3}{S} \cdot \frac{f_3}{s_3}, \ldots \frac{s_n}{S} \cdot \frac{f_n}{s_n}$$

represent respectively: the firsts the probabilities of the event relative to the hypotheses $h_2, h_3, \ldots h_n$; the second the probabilities of the concurrence of each of these hypotheses and of the event; that is the probabilities that the event will take place by virtue of the hypothesis h bearing the same n° as the numbers s and f.

We suppose now that the event is certain, or else that it has happened, then one of F cases which are favorable to it has taken place, the probability that this case is comprehended among those which favor a hypothesis h_i and which are in the number of f_i are will be evidently

$$\frac{f_i}{F}.$$

This is the sought probability, that which results from the even for the hypothesis h_i. It is acceptable to present this probability under the form that one is able to calculate immediately according to the givens of the question; for that we have only to replace f_i by

$$\frac{s_i f_i}{s_i}$$

and F, or

$$f_1 + f_2 + f_3 + \cdots + f_n,$$

by

$$\frac{s_1 f_1}{s_1} + \frac{s_2 f_2}{s_2} + \frac{s_3 f_3}{s_3} + \cdots + \frac{s_n f_n}{s_n}$$

next to divide the top and the bottom by S, this which will give us

$$\frac{\frac{s_i}{S}\frac{f_i}{s_i}}{\frac{s_1}{S}\frac{f_1}{s_1} + \frac{s_2}{S}\frac{f_2}{s_2} + \frac{s_3}{S}\frac{f_3}{s_3} + \cdots + \frac{s_n}{S}\frac{f_n}{s_n}}$$

or by making for brevity usage of the summation sign $\sum$

$$\frac{\frac{s_i}{S}\frac{f_i}{s_i}}{\sum \frac{s}{S}\frac{f}{s}}$$

Thus the probability of a hypothesis that one will have made in order to explicate a certain event, or even already arrived, is equal to the product of the probability of the hypothesis taken in itself, or independently of the event and of the probability of the event, by supposing the hypothesis certain, this product being divided by the sum of the similar products relative to all the hypotheses.

If among the numbers f there is found of them which are zero, or that which reverts to the same if some hypotheses h furnish no chance to the event, one could separate these hypotheses as if they did not exist. In fact we suppose that the numbers f departing from f_m are zeros, that is

$$f_{m+1} = 0, \quad f_{m+2} = 0, \ldots f_n = 0.$$

It is clear first that the probabilities of the hypotheses corresponding to these numbers are zero, nest the probability of each other hypothesis h_i will be

$$\frac{\frac{s_i}{S}\frac{f_i}{s_i}}{\frac{s_1}{S}\frac{f_1}{s_1} + \frac{s_2}{S}\frac{f_2}{s_2} + \frac{s_3}{S}\frac{f_3}{s_3} + \cdots + \frac{s_m}{S}\frac{f_m}{s_m}};$$

now the number S by vanishing of itself from this expression one will be able to replace it by a number at will, for example by the sum

$$s_1 + s_2 + s_3 + \cdots + s_n$$

where the s in the superior N^{os} to m are not found.

Thus in the calculation of the chances s_1, s_2, s_3, $\ldots s_m$ favorable to the different hypotheses h_1, h_2, h_3, $\ldots h_n$ one is able to have regard only to the hypotheses h_1, h_2, h_3, $\ldots h_m$ which contain the chances favorable to the arrival of the event and to reject all the others

$$h_{m+1} \quad h_{m+2}, \ldots h_n$$

which do not contain it, whatever be besides the numbers s_{m+1}, s_{m+2}, $\ldots s_m$ of the favorable chances which are proper to them.

As the expression of the probability of a hypothesis h contains only the ratios among the numbers S, s and f, one could make two counts of the equally possible chances: one by seeking the numbers s, thus also their sum S, finally to employ them in the ratios $\frac{s}{S}$; another by calculating the same numbers s and the numbers f in order to be served in the ratios $\frac{f}{S}$. It is able that one will find for s some different values in the two counts, for in the first one will consider only the hypotheses alone but all simultaneously, and in the second one will consider each hypothesis in particular and the event. In truth one would be able to render the values equals of which there is concern by the subdivision of the chances, but this subdivision is superfluous, it will do only to lengthen the calculation. For the rest the preceding remark is itself superfluous because of its evidence, for it concerns not the chances alone, but of the probabilities as much of the hypotheses as of the event, thus it will make only that which is necessary in order to have the probabilities.

3. We are going now to present some observations on the analysis of Laplace. The illustrious geometer has considered the question under two points of view. In the first he puts without demonstration the principle which follows and which we copy verbatim.

Principle

> "If an event is able to be produced by a number of different causes, the probabilities of the existence of thes causes taken from the event, are among them as the probabilities of the event taken from these causes, and the probability of the existence of each of them, is equal to the probability of the event taken from this cause divided by the sum of all the probabilities of the event taken from each of these causes."

Laplace will consider only the particular case when the hypotheses are equally possible a priori, and one shows that he admits as principle the proportionality between the probabilities of these hypotheses, drawn from the event and the probabilities of the event drawn from the hypotheses. This principle is exact, but it had been necessary to establish the exactitude before making usage of it, that which the illustrious geometer has not done.

We have seen that the probability of the hypothesis h_i following the event and that of the event according to the hypothesis h_i were respectively

$$\frac{f_i}{F} \text{ and } \frac{f_i}{s_i};$$

their ratio will be

$$\frac{s_i}{F}.$$

Now in the particular case of the causes a priori equally possible the quantities s_1, s_2, s_3, $\dots s_n$ will be equals among them, of which the preceding ratio will not change by passing from one hypothesis to another, this which reverts to the proportionality admitted by Laplace. Gauss has rigorously demonstrated this proportionality and we have done only to apply his analysis to the case of the causes a priori unequally possible, a case that this illustrious geometer has not considered.

Laplace in his *Théorie Analytique des Probabilités* considers the principle under another point of view, he admits the inequality between the product of the probability of the event a priori, by that of a hypothesis following the event, and the product of the probability of the same hypothesis a priori by that of the event according to the hypothesis. It is easy to be assured that this inequality, admitted by Laplace without demonstration, subsists in fact. The probability of the event a priori and that of a hypothesis h_i following the event are

$$\frac{F}{S} \text{ and } \frac{f_i}{F};$$

thus their product will be

$$\frac{f_i}{S};$$

on the other hand, the probability a priori of the hypothesis h_i, and that of the event according to this hypothesis being

$$\frac{s_i}{S} \text{ and } \frac{f_i}{s_i}$$

will give the same product

$$\frac{f_i}{S}$$

as the one which precedes. But he did not act to verify the principle by the value obtained for the unknown, it would be necessary on the contrary to be served of the principle for the determination of the unknown. Moreover it can be that the principle in question was for Laplace of an entire evidence and would require no demonstration, as for us, we swear that it would not appear to have this degree of evidence.

Moreover Poisson has demonstrated it in his researches on the probability of judgments. He establishes first for the particular case of the hypotheses a priori, equally probable,[1] next he treats the general case,[2], but the considerations of which he has made usage, exact without doubt, do not appear to us completely direct.

[1] *Récherches sur la probabilité des jugements*. Page 81 and following.
[2] Ibid. Page 93 and following.