FORMULES DE PROBABILITÉS

RELATIVE

AU RÉSULTAT MOYEN DES OBSERVATIONS,

QUI PEUVENT ÊTRE UTILES DANS L'ARTILLERIE.*

Article communicated by Mr. Poisson,
member of the Institut, examiner of the Artillery.†

Mémorial de l'artillerie, No. III (1830), pp. 141–156.

When measuring many times any one thing that we will call A, one has obtained some equal or unequal values, and when one has no reason to prefer the one to the other, it is evident that one must take the *mean* of all these values, or their sum divided by the number of measures, for the value of A which results from these observations. One is naturally carried to think that this mean value will approach more and more the true value of A, in measure as one will increase the number of observations; but the calculus of probabilities is able alone to furnish the means to evaluate with precision the error that one will have to fear, by making us know the limits of this error, and their degree of probability. Lagrange is the first who has submitted to the calculus this important question, and who has stated the principles on which it depends; but the solution that he as given of it would not know how to be useful in practice, because it supposes known the law of probability of the errors of experience, and that this law is, on the contrary, completely unknown; its representation by a function of the variable magnitude of the errors being, to speak properly, only a sort of necessary fiction in order to give a hold and to serve as base to the mathematical analysis. It is to Laplace that the sciences of observation are indebted of a rule independent of every law of probability of the deviations of experience, applicable to all the cases where the number of observations is considerable, and in which one makes use only of the values observed of the unknown. Here is in what this rule consists, of which the author has made numerous applications in his *Théorie analytique des probabilités*.

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†Mr. Poisson having noted, by the examination of the memoirs sent to the competition, that the usage of the formulas relative to the calculus of probabilities was not very familiar to the officers of the artillery, has believed a useful thing to do by writing this note, in which he has interested himself in presenting these formulas with as much concision as possible, by indicating in them as example some applications to the artillery, and especially to fix the precise sense that one must apply there.

Let n be the number of measures that one has made. We designate by a_1, a_2, a_3, $\ldots a_n$, the n equal or unequal values that they have given for A. Let s be their sum and m the mean of all these values, so that one has

$$\begin{aligned} s &= a_1 + a_2 + a_3 + \cdots + a_n, \\ m &= \frac{s}{n}. \end{aligned}$$

If one subtracts successively m from the observed values $a_1, a_2, a_3, \ldots a_n$, one will have their deviations on both sides from the mean, which will be in general very small quantities by which one has custom to judge of the goodness of the observations. We call h the half sum of their squares, that is we make:

$$2h = (a_1 - m)^2 + (a_2 - m)^2 + \cdots + (a_n - m)^2 \ldots \quad (1)$$

There will be a certain probability p that the error to fear in taking m for the value of A, will be comprehended between the limits:

$$\pm \frac{2\alpha\sqrt{h}}{n},$$

or, in other words, if one calls x the true value of A, which is unknown, there will be this probability p that the difference $x - m$ will not surpass $\frac{2\alpha\sqrt{h}}{n}$, setting aside the sign; α being a numerical coefficient that one will take at will and from where the value of p will depend. Now, if n is a large number, and if one neglects the quantities of the order of the fraction $\frac{1}{n}$, the value of p will be given by the equation

$$p = \frac{2}{\sqrt{\pi}} \int_0^\alpha e^{-t^2} dt, \ldots \quad (2)$$

in which e is the base of the Naperian logarithms, and π the ratio of the circumference to the diameter. Following the use received for some time, the sign $\int_0^\alpha$ indicates a definite integral, taken from zero to α.

If one wishes to have $p = \frac{1}{2}$, it will be necessary to $\alpha = 0.4764$ very nearly. Thus there will be a probability $\frac{1}{2}$ or odds one against one, that the error to fear concerning the mean of a great number n of observations will fall between the limits:

$$\pm \frac{0.9528}{n} \sqrt{h};$$

the value of h being deduced from their deviations, by means of the equation (1).

One finds at the goal of the analysis of astronomical refractions of Kramp, a table of numerical values of the integral

$$\int^\alpha e^{-t^2} dt,$$

whence one will conclude immediately those of the integral that equation (2) contains, by observing that

$$\int_0^\alpha e^{-t^2} dt = \frac{1}{2}\sqrt{\pi} - \int_{-\alpha}^\infty e^{-t^2} dt.$$

According to this table, the value of p approaches very rapidly to unity in measure as α increases; and, for a very considerable value of α, that of p differs very little from unity or from certitude. Here are the values of p which correspond to some values of 2α: one has

$$
\begin{aligned}
2\alpha =&0.5 \cdots \ p =0.2764;\\
2\alpha =&1.0 \cdots \ p =0.5205;\\
2\alpha =&1.5 \cdots \ p =0.7112;\\
2\alpha =&2.0 \cdots \ p =0.8426;\\
2\alpha =&2.5 \cdots \ p =0.9229;\\
2\alpha =&3.0 \cdots \ p =0.9661;\\
2\alpha =&3.5 \cdots \ p =0.9868;\\
2\alpha =&4.0 \cdots \ p =0.9953;\\
2\alpha =&4.5 \cdots \ p =0.99854;\\
2\alpha =&5.0 \cdots \ p =0.99959.
\end{aligned}
$$

One will obtain the corresponding limits of the error to fear concerning of the mean result of the observations, by dividing the values of 2α by the number n of measures; and multiplying in each case, by the square root of the quantity h, calculated by means of equation (1).

This square root will increase, in general, with the number n, but in a lesser ratio; and one demonstrates that the fraction $\frac{\sqrt{h}}{n}$ is very nearly equivalent to another fraction $\frac{g}{\sqrt{n}}$, of which the numerator g does not depend on n. There results that the quantity α remaining the same, and consequently also the probability p, the limits of the error of the mean result will be tightened more and more, and this result will converge indefinitely toward the true value of A, in measure as one will employ a greater number of observations. Reciprocally, if one makes α vary with the number n, and if one takes it proportional to its square root, the limits of the error will be constants, but the probability p will increase more and more, at the same time as the number n; that is, it will be more and more probable that the error to fear concerning the mean result will fall between some given limits, and that one will be able to render this probability as close to certitude as one will wish, by increasing conveniently the number of observations.

The preceding formulas will be so much more exact, as the number n will be greater. One will be able further to use them with utility in the artillery, when one will have only ten or twelve measures on one same thing A; but for the lesser number of observations, the calculus of the probabilities furnishes no longer any means to evaluate the limits of the error to fear, by taking their mean result for the value of A. It is only by a very attentive examination of the causes of errors which are able to influence over the experiences, that one will judge in each particular case, the degree of confidence of which this mean value will be susceptible.

These formulas suppose essentially that there is in the instruments that one has employed, no more than in the circumstances which have been able to influence on the measures, any *constant* cause of error, that is, no cause which renders the errors rather preponderant in one sense than in another. It is especially against the constant causes of error that the observers must be forearmed. If there exist them, the mean result will no longer converge toward the true value of the thing one wishes to know; and, whatever be the number of observations, they would be deceptive, and it would be necessary to reject it. But, on the other hand, if it is indispensable that in each observation the equal errors, the one to more and the other to less, are equally probable, on the one hand it is not necessary that the law of probability of errors is the same for all the observations; and this unknown law is able to change arbitrarily from one observation to another.[1] Thus, for example, an angle having been measured, a body having been weighed a great number of times, with some different instruments and by different persons, and the condition of the equality of chance of the equal errors and of contrary signs being supposed to hold, the formulas will make known the probability of the error to fear, by taking for that angle or for that weight the mean of all the values found.

If one applies successively these formulas to two series, one of n and the other of n' observations, of which the mean results are respectively m and m', and if one designates by k^2 and by k'^2 the corresponding values of n, calculated according to equation (1), there will be the same probability p, given by equation (2), that the difference $x-m$ will be less than $\frac{2\alpha k}{n}$, and that the difference $x-m'$ will be smaller than $\frac{2\alpha k'}{n'}$; x being the true value of the thing A that one has measured, which unknown value is the same for these two series. The observations, in order to be good, must give values of k and k' very small, with respect to the numbers n and n', or very small values of the fractions $\frac{k}{n}$ and $\frac{k'}{n'}$. Between the two different series, the better, in equal number, will be that which will give the lesser value for the second member of equation (2). The differences $x - m$ and $x - m'$ being very small according to their probable limits, it will be likewise in regard to $m - m'$; consequently, if one measures many times one same thing by similar or different means, the means of the series composed of a great number of observations, will differ very little among them; and if that not happen, one will be thence right to conclude from it that this thing has changed in magnitude, or that there is in one of the series of observations some constant causes of errors, which render from them the consequences faulty and put the formulas at fault.

The rules that one just reported agree also with the case where the differences among the values of A, given by experience, arise not from errors of observation, but are due to some accidental physical causes. Each value found for A is then a true value, resulting from these causes; but among all the values of which this thing is susceptible, one supposes that there exists one which is such, that two equal deviations, one to the plus and the other to the less, are equally probable in each observation: it is this value, thus defined, that one regards as the true value of A, and of which the mean result of a great number of observations approaches more and more in measure as this number increases. We suppose, for example, that A is the range of a gun of which the caliber and the length are determined, at the same time as the weight of the charge and the quality of the powder, shooting under a given angle and at a height above the ground also given. There will be no error out of the measure of A at each shot; however the measured magnitudes of A will be able to differ from one shot to another, by reason of accidental causes, as from small inequalities in the force or the weight of the powder, in the position of the bullet, in the manner of which one will have crammed, etc., which will influence so much more, so much less on the magnitude of the range; now, one supposes tacitly that there exists an unknown range x, such that the probability of deviating from it at each shot is the same on this side and on the other; and it is

[1] Additions to *Connaissance des Tems* of 1832, p. 3.

this unknown x that one proposes to determine, at least by approximation, by means of a series of observations. It is not necessary that they have taken place in the same period; the principal circumstances whence depend the length of the range being the same, one is able to arrange the influence of such or such gun, if it exists, and the influence of the temperature and of the density of the air, in some different periods, among the accidental causes which make the range vary from one shot to another, to more or to less, and which balance themselves on a great number of shots. One will be able therefore to make agree in the determination of which there is concern the measured ranges in the different schools during many years; if one designates by m the sum of all these ranges, divided by their number n, there will be the probability p given by equation (2), that the difference $x - m$ will not surpass $\frac{2\alpha\sqrt{h}}{n}$; the value of h being calculated by means of equation (1). By taking $2\alpha = 5$, for example, this probability will be equal to 0.99959, and will not differ sensibly from unity which represents certitude; if therefore the fraction $\frac{\sqrt{h}}{n}$ is very small, one will be able to take the mean m for the value of x, without fear of error greater than nine times this fraction, to more or to less.

Knowledge of the range furnishes, as one knows, a means to calculate the muzzle velocity of the bullet which corresponds to a given charge. It is necessary for that to make use of the horizontal shot, that which permits expressing very simply the muzzle velocity as a function of the range,[2] for which one will take the approximate value m, determined as one just said. In truth, this function of the range contains also the coefficient of the resistance of air proportional to the square of the velocity, on which there is able to remain some doubt. But by employing successively the mean ranges of two series of observations, which will differ from one another by the height of the point of departure of the bullet above the point of fall, and in which all the other elements will be the same, one will have two expressions of the muzzle velocity which must be equals; whence there will result one equation of which one will be able to serve oneself in order to determine the numerical value of the coefficient of the resistance, or in order to verify that which is most generally admitted. It would be interesting to know if this coefficient varies sensibly with the temperature of the air, as well as with the density; the means that one indicates in order to obtain from it the value would have the advantage of not prejudicing the question to know if the muzzle velocity depends on the inclination of the gun, since one changes the mean range from one series to another, only by the elevation of the gun above the ground.

The firing to the *target* presents again an useful application of the preceding formulas. We suppose that the target is circular and of a great enough diameter in order to be attained in all the shots; we suppose also that the soldier aims constantly and the best that he is able toward the center of this disk, and we draw through this point two straight lines, the one horizontal and the other vertical. Independently of the errors in the direction of the shot, which will be so much in one sense and so much in another, there will be able to exist in the construction even of the rifle a constant cause which will tend to raise or to lower the shots, and another constant cause which will tend to carry them toward the right or toward the left. This being, there will be some unknown coordinates x and y from a point of the target, returned to the two axes drawn through

[2] *Traité de méchanique* of Mr. Poisson, Tome I, pag. 351.

its center, which will be such, that at each shot one would be able to wager in an equal game that the abscissa of the point where the ball will come to strike will surpass x or will be less, and also in an equal game, that its ordinate will be greater or smaller than y. One will measure with care the distances to the vertical axis of the points where the balls will have touched; one will make the sum of them, and regarding as positives the distances of the points situated on one side of the axis, and as negatives those of the points situated on the other side: if the number of shots is considerable, that sum divided by this number will be an approximate value of the unknown x. By operating likewise in regard to the distances to the horizontal axis, one will obtain an approximate value of y. In order that the rifle be *just*, that is, on order that it have no tendency to deviate to right or to left, or else to lower or to elevate the shots, it will be necessary that these two values are both null or negligible. One must reject the arm which will not satisfy these conditions; but two rifles which will fulfill them will not be for this equally good: if one makes, for each rifle, the sum of the squares of the distances of the balls to the center of the target, and if one divides this sum by the square of the number of shots, the better rifle will be the one for which this quotient will be least, the number of shots being very great on both sides. It is useless to say that in this comparison, the shots must be drawn at the same distance from the target, with equal charge, and in same proportion by the different soldiers for the two rifles. One soldier will be more skilled than the other, if with the same arm, with equal charge and at the same distance, the quotient of which there is concern is less for the first soldier than for the second. It will be able that one same rifle is not equally good, at different distances from the target, and for unequal charges. It will be better at the distance, and for the charge which would correspond to the smallest values of this same quotient.

In order to know the probability to attain a target of a given radius with a rifle known just, or for which the values of the coordinates x and y, concluded from a great number of shots, will be null or insensible, we designate by f^2 the sum of the squares of the distances of these shots to the center of another target, of a diameter great enough in order to comprehend them all, by n their number, by r the radius of the given target; we determine the quantity α by means of the equation:

$$\frac{2\alpha f}{n} = r;$$

The probability to touch this target will be the value of p given by equation (2), and corresponding to that value of α: if one knew only that, out of a great number n of tests, one has touched a target or attained any goal, a number of times represented by μ, the probability to touch this same goal, with the same goal, with the same arm and at the same distance, would be expressed by the ratio $\frac{\mu}{n}$, so that, if one would fire anew a single shot, there would be odds μ against $n-\mu$ that the shot would carry, and, if one fired a new series of a very great number of shots n', it would be very nearly certain that the ratio of the number of shots which would touch, to that number n, would differ very little from the fraction $\frac{\mu}{n}$. This rule is deduced from the probability of future events, concluded from the observation of past events; that it is not necessary to confound with that which corresponds to the probability of the mean result of the observations. It is able to suffice in the case of the shot of the bullet or of the bomb; but it seems that, in order to judge the justice of small arms, and in order to compare them among themselves, it would be useful to recur to the preceding rules, where one make enter, not only the number of times that the goal has been attained, but yet the deviations more or less great of the different shots.