MÉMOIRE

SUR LA

PROBABILITÉ DU TIR A LA CIBLE*

Mr. Poisson

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Following from the *formules de probabilités* inserted into n° III of the *Mémorial de l'artillerie*, I have indicated, in a few words, their usage in the case of target shooting. I myself propose now developing, in a convenient manner, this application of the calculus of probabilities, of which Messers the officers of the artillery have recognized the utility, and to which they have had recourse in order to compare different rifles under the report of the justness of the shot. One will find in the *Théorie analytique des probabilités* of Laplace, or, if one wishes, in my preceding memoirs on this matter,[1] the demonstrations of the formulas which I will suppose known.

(1) The *probability* of an event is the reason that we have to believe that this event will arrive or is arrived. One demonstrates that it has for measure the ratio of the number of favorable cases to the event, to the number of all the possible cases. In this evaluation, it is necessary that all the cases, favorable or contrary, are equally possible, not in themselves, but according to that which we know concerning them; whence there results that the probability of one same event will be able to be very different for two persons who will not have the same the same knowledge concerning this event. The person who will have the most knowledge will carry a more enlightened judgment; and although this judgment is based sometimes on a lesser probability than that which another person will have determined, it is however the opinion of the first which will always be reasonable to adopt.

Two persons know, for example, that an urn A contains five white balls and on black ball, and an urn B, three white balls and four black balls. One draws a ball from one of these two urns. One of the two persons do not know from which urn the ball has been extracted. For it, according to the simplest rules of the calculus of chances, the probability that the ball is white and exited from A, has for value the product of $\frac{5}{6}$ and of $\frac{1}{2}$; the probability that it is white and drawn from B, has likewise for measure the product of $\frac{3}{7}$ and of $\frac{1}{2}$; consequently, the probability that the ball is white, has for total value $\frac{5}{12} + \frac{3}{14}$, or $\frac{53}{84}$; and this fraction surpasses $\frac{1}{2}$, the opinion of the first person must be that the ball is white. On the contrary, the other person knows moreover that the

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[1] *Connaissance des Tems*, for the years 1827 and 1832. *Mémoires de l'Académie des Sciences*, Tome IX.

ball is extracted from B; for him the probability that the ball is black will be therefore $\frac{4}{7}$, and this must be also his opinion. One must adopt his judgment, since this second person knew all that which was known of the first, and that he knew, besides, the urn from which the ball has exited. However, his opinion is based on the probability $\frac{4}{7}$ or $\frac{48}{84}$, less than the probability $\frac{53}{84}$ on which the contrary opinion was based.

This circumstance of two judgments contrary and more or less based, is able to be presented in the comparisons that one makes of different rifles, in order to judge of their justice according to target shooting. We suppose, for example, that a soldier has attained the target 50 times out of 100, and that with another rifle, the same soldier, or a soldier to which one supposes the same skill, has attained the target 60 times out of 100; if one has no other data, one will judge the second rifle preferable to the first. But we suppose that one is not limited to count the number of shots which are fired in the two experiences, and one has measured, besides, the distances to the center of the target, of all the points where it has been attained; it will be able to happen that by having regard to these distances, the first rifle is judged more just than the second; and, in all cases, it will be this second opinion contrary or similar to the first, that it will be necessary to adopt, since it is based on the data of the first judgment, and on some other data of which one had not first taken account. One is going to exposed the rules that the calculus of probabilities furnishes for the comparison of the justice of the arms or of the skill of the soldiers, when one has regard to the distances and to the distribution of the shots on the plane of the target.

(2). I will call C the center of the target. Through this point, I draw two straight lines, the one horizontal and the other vertical. I take these straight lines for axes of the coordinates; and I designate by x and y the coordinates of any point M of the target; so that x is the distance from M to the vertical axis, that one will regard as positive or as negative, according as M will be situated to right or to left of this axis, and that y is the distance from M to the horizontal axis, also positive or negative, according as M will be found above or below this straight line.

I represent by Fx the probability that the center of the ball will come to strike the target between the vertical axis of y and the parallel to this axis drawn through the point M, and by $F'y$ the probability that the center of the ball will strike between the horizontal axis of x and a parallel to this axis also drawn through the point M. The probability that the center of the ball will come to strike between the axis of y and another vertical, drawn at the distance x' from this axis, will be likewise Fx', and, consequently, the difference $Fx' - Fx$ will express the probability that this center will strike between the two drawn verticals drawn at the distances x and x' from the axis of y. If the difference $x - x'$ is infinitely small and equal to dx, this probability will be also infinitely small and equal to $\frac{Fdx}{dx}dx$, or to $fxdx$, by making $\frac{dFx}{dx} = fx$. This product $fxdx$ will be therefore the probability that the center of the ball will attain the vertical drawn through the point M. By making $\frac{dF'y}{dy} = f'y$, the product $f'ydy$, will be likewise the infinitely small probability that the center of the ball will attain the horizontal straight line drawn through the same point. The product of $fxdx$ and $f'ydy$, or $fxf'ydxdy$ will express therefore the infinitely small probability of the second order, that the center of the ball will come to fall precisely at the point M, or on the differential element $dxdy$ of the plane of the target.

The functions fx and $f'x$ are unknown to us. If there is no constant cause in the construction of the rifle or in the manner of aim of the soldier, which makes the shots deviate rather from one side than from the other of the vertical drawn through the point C, or which tends to raise them or to lower them above or below the horizontal straight line passing through this point, it is evident that fx and $f'y$ must express the laws of probabilities similar on the two sides of this axis, and that one must have $f(-x) = fx$ and $f'(-y) = f'y$, whatever be the variables x and y. This is that which one does not know in advance, and that the tests will make known, as one will see in the following. Each of these functions fx and $f'y$ will be a positive quantity, which will become sensibly null for every value of the variable great enough, setting aside the sign, in order to render immediately improbable the deviation expressed by this value. In every case, one will have

$$\int_{-\infty}^{\infty} fx dx = 1, \qquad \int_{-\infty}^{\infty} f'y dy = 1,$$

since the limits of these integrals comprehend all the distances to each axis from the coordinates, where the little ball attains the plane of the indefinitely prolonged plane of the target.

This granted, it will suffice to consider the probability of the deviations on both sides of the axis of y; some formulas similar to those that we are going to write, will hold relative to the axis of x.

(3) We represent by ϕ and ρ some functions of a variable z, determined by these equations:

$$\begin{aligned} \int_{-\infty}^{\infty} fx \, \cos zx \, dx &= \rho \cos \phi \\ \int_{-\infty}^{\infty} fx \, \sin zx \, dx &= \rho \sin \phi \end{aligned}$$

One will have, at the same time,

$$\rho^2 = \left(\int_{-\infty}^{\infty} fx \, \cos zx \, dx \right)^2 + \left(\int_{-\infty}^{\infty} fx \, \sin zx \, dx \right)^2,$$

the tangent of the angle ϕ will be the ratio of the two definite integrals; this angle will be real; it will be able to extend from zero to $360°$, that which will permit to consider ρ as a positive quantity.

We designate by s the sum of the distances x, positives or negatives, which will have place in any number n of shots; and we call V the probability that this sum s will be comprehended between some given limits that one will represent by $b - c$ and $b + c$, by designating by c a positive quantity, and by b a positive or negative quantity. One demonstrates, in the calculus of probabilities, that one will have

$$V = \frac{2}{\pi} \int_0^{\infty} \rho^2 \cos(n\phi - bz) \frac{\sin cz}{z} dz. \tag{1}$$

The value of ρ^2 is able to be written under this form:

$$\rho^2 = \int_{-\infty}^{\infty} fx \cos zx \, dx \int_{-\infty}^{\infty} fx' \cos zx' \, dx' + \int_{-\infty}^{\infty} fx \sin zx \, dx \int_{-\infty}^{\infty} fx' \sin zx' \, dx',$$

or, that which is the same thing, under this here;

$$\rho^2 = \int_{-\infty}^{\infty} int_{-\infty}^{\infty} fxf'x \cos z(x - x') \, dxdx';$$

whence one concludes that for $z = 0$, one will have $\rho = 1$, and that for each value of z which will be null, the quantity ρ will be less than unity. It is according to this consideration that one will be able to transform formula (1), and conclude from it an approximate value, but much simpler of V, when n will be a very great number.

In this case, the factor ρ^2 contained under the sign $\int$, will be extremely small, since z will have a value which will be very small; it will suffice therefore to understand the integral relative to z, from zero to a very small value of this variable; and, that being, one will be able to develop the functions ρ and ϕ into very convergent series, ordered according to the powers of z. By making, for brevity,

$$\int_{-\infty}^{\infty} xfx \, dx = k, \qquad \int_{-\infty}^{\infty} x^2 fx \, dx = l,$$

and neglecting the powers of z superior to the second, one will have

$$\rho = 1 - \frac{1}{2}(l - k^2)z^2, \qquad \phi = kz$$

The quantity $l - k^2$ depending on the form of the function fx, its value is not known to us *a priori*; but one is able to demonstrate that it is always positive; that which is besides necessary in order that ρ be less than unity.

In fact, because of $\int_{-\infty}^{\infty} fx \, dx = 1$, one has

$$l - k^2 = \int_{-\infty}^{\infty} x^2 fx \, dx \int_{-\infty}^{\infty} fx' \, dx' - \int_{-\infty}^{\infty} xfx \, dx \int_{-\infty}^{\infty} x'fx' \, dx',$$

or, that which is the same thing,

$$l - k^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x^2 - xx') fxfx' \, dxdx';$$

one has equally

$$l - k^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x'^2 - xx') fxfx' \, dxdx';$$

and by taking the half-sum of these two values, there comes

$$l - k^2 = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - x')^2 fxfx' \, dxdx';$$

a quantity evidently positive, and which will not be reduced to zero.

This being, I make

$$\frac{1}{2}(l-k^2)=g^2;$$

there will result from it

$$\rho^2=(1-g^2z^2)^2;$$

by developing by the formula of the binomial, and setting next n in the place of $n-1$, $n-2$, $n-3$, etc., we will have

$$\rho^2=1-ng^2z^2+\frac{1}{2}n^2g^4z^4-\frac{1}{2.3}n^3g^6z^6+\text{etc.},$$

or, that which is the same thing,

$$\rho^2=e^{-ng^2z^2};$$

and according to this value of ρ^2 and that of ϕ, formula (1) will become

$$V=\frac{2}{\pi}\int e^{-ng^2z^2}\cos(nk-b)z\sin cz\frac{dz}{z}.$$

For each value of z which is not very small, the exponential factor contained under the $\int$ sign is insensible because of the magnitude of the number n; although z be a positive variable that one has supposed very small, one is able therefore now, without sensible error, to extend the integral from $z=0$ to a value of z as great as one will wish, and event to $z=\infty$. By taking, besides,

$$b=nk,$$

we will have

$$V=\frac{2}{\pi}\int_0^\infty\int_0^c e^{-ng^2z^2}\sin cz\frac{dz}{z};$$

a result that one is able to write thus

$$V=\frac{2}{\pi}\int_0^\infty\int_0^c e^{-ng^2z^2}\cos\nu z\,dzd\nu,$$

because one has identically

$$\int_0^c\cos\nu z\,dv=\frac{1}{2}\sin cz.$$

But according to a known formula, one has

$$\int_0^\infty e^{-ng^2z^2}\cos\nu z\,dz=\frac{\sqrt{i}}{2g\sqrt{n}}e^{-\frac{\nu^2}{4ng^2}}.$$

By reversing the order of integrations relative to ν and to z, one will have therefore

$$V=\frac{1}{g\sqrt{\pi n}}\int_0^c e^{-\frac{\nu^2}{4ng^2}}d\nu.$$

I make next

$$c = 2\alpha g\sqrt{n}, \quad \nu = 2tg\sqrt{n}, \quad d\nu = 2g\sqrt{n}dt;$$

the limits relative to the new variable t will be $t = 0$ and $t = \alpha$; and one will have

$$V = \frac{2}{\sqrt{\pi}}\int_0^\alpha e^{-t^2}dt,$$

or finally

$$V = 1 - \frac{2}{\sqrt{\pi}}\int_\alpha^\infty e^{-t^2}dt, \tag{2}$$

for the probability that the sum s of a very great number n of values of x, positives or negatives, will be comprehended between the limits $b \pm c$, that is between the limits

$$nk \pm 2\alpha g\sqrt{n},$$

according to the values that one has taken for b and c.[2]

If s' is the sum of the n values of y, and if one makes

$$\int_{-\infty}^{\infty} yf'ydy = k', \quad \int_{-\infty}^{\infty} y^2 f'ydy = l', \quad \frac{1}{2}(l' - k'^2) = g'^2,$$

formula (2) will express also the probability that this sum s' is comprehended between the limits

$$nk' \pm 2\alpha g'\sqrt{n}.$$

(4) Instead of the sums s and s' of the n values of x and y, one is able to consider the means of these values; by designating them by λ and λ', so that one has

$$\lambda = \frac{s}{n}, \qquad \lambda' = \frac{s'}{n'},$$

and dividing by n the preceding limits, there will result from it

$$k \pm \frac{2\alpha g}{\sqrt{n}}, \quad k' \pm \frac{2\alpha g}{\sqrt{n'}},$$

for the limits of λ and of λ', which will have, both, a probability V expressed by formula (2).

In giving to α a great value, one will render very small the value of the integral contained in formula (2), and the value of V, sensibly equal to unity, so that the preceding limits of λ and λ' will be also very nearly certain. In order to fix the ideas, if one takes $\alpha = 1.8$, there will be, very nearly, odds of one hundred against one that the limits of λ will be $k \pm \frac{3.6g}{\sqrt{n}}$, and the same probability that λ' will fall between the limits $k' \pm \frac{3.6g'}{\sqrt{n'}}$. The quantities g and g' being independent of n, it follows that when this number will increase more and more, the limits will tighten further, and the values of λ and λ' will differ less and less, from k and k'. Reciprocally, if one has measured the

[2]One finds at the end of the *Réfractions astronomiques* of Kramp, a table of the numerical values of the integral contained in formula (2),which extends to the values of k, from $k = 0$ to $k = \infty$.

values of x and of y which correspond to a very great number n of shots, and if one takes the means λ and λ' of these values by having regard to their signs, the quantities λ and λ' would be able to be regarded as equals to the unknowns k and k', from which they deviate so much less as the square root of n will be a greater number. Now, the integral $\int_{-\infty}^{\infty} xfx\,dx$, that k represents, is evidently null, when one has $f(-x) = fx$, that is when the equal deviations are equally probable on both sides of the vertical drawn through the center C of the target; consequently, when there exists no constant cause which is able to make the shots deviate rather on one side than on the other of this straight line, one must find the mean λ of a very great number of distances x, a value null or negligible; and likewise if there is no constant cause which tends to raise or lower the shots above or below the horizontal straight line drawn through the point C, it follows from the integral $\int_{-\infty}^{\infty} yf'y\,dy$ represented by k', that one must also find for the mean λ' a value sensibly null. On the contrary, when one will obtain for λ or λ' a value which is not negligible, one will conclude from it that it exists, either in the construction of the arm, or in the manner of aiming of the soldier, a constant cause of deviation. By making the tests recommence with many other soldiers, if one finds always for λ or for λ' some notable values, there will be place to believe that the constant cause proceeds from the arm, which will not be then a *just* rifle, and that one must reject, if one judges that the mean deviations k and k', or their approximate values λ and λ', are too considerable.

When the quantity k is null for two different rifles, the better is evidently the one for which the quantity g is smallest; for it is the one which leaves to fear, in a great number of shots, and with one same probability, the lesser mean deviation on one part or the other of the vertical drawn through the point C. Likewise, between two rifles for which the quantity k' is zero, one must prefer the one which corresponds to the lesser value of g', and consequently, to the most tightened limits of the mean deviation, on one part or the other of the horizontal straight line passing through the same point. When these constants g and g' are unequal, one same rifle is able therefore to be preferable under the relation of the horizontal deviations, and the less good relative to the deviations in the vertical sense. In all cases, these two quantities are not known to us *a priori*, since they depend on the unknown functions fx and $f'y$; but one is able, as for k and k', to determine the very probable and very near values of g and g', deduced from observations made in very great number. One arrives by a quite simple consideration, of which Laplace has first made use, and that I am going to expose.

(5) The square of the distance from point M to the axis of y is x^2, either when this distance is $+x$, or when it is $-x$; consequently if one gives to x only the positive values, and if one designates by $f_{\prime}xdx$ the infinitely small probability that the square of the distance to this axis, from the point where the ball will come to strike the target, will be equal to x^2, one will have

$$f_{\prime}x = fx + f(-x).$$

This being, I represent by $s_{\prime}$ the sum of the squares of the distances from the center of the ball to that same axis, which will hold in any number n of shots; I set x^2 and $f_{\prime}xdx$ in the place of x and of $fxdx$ in the quantities ρ and ϕ of n° 3; and I extend only the integrals to the positive values of x, so that one has

$$\rho\cos\phi = \int_0^\infty f_{\prime}x\cos zx^2\,dx,$$
$$\rho\sin\phi = \int_0^\infty f_{\prime}x\sin zx^2\,dx:$$

formula (1) will express the probability V that the sum $s_{\prime}$ will be comprehended between the given limits $b \pm c$. Therefore by passing, as previously, to the case where n is a very great number, and making, for brevity,

$$\int_0^\infty x^2 f_{\prime}x\,dx = k_{\prime}, \quad \int_0^\infty x^4 f_{\prime}x\,dx = l, \quad \frac{1}{2}(l_{\prime} - k_{\prime}^2) = g_{\prime}^2,$$

formula (2) will be the probability that the sum $s_{\prime}$ will have for limits

$$nk_{\prime} \pm 2\alpha g_{\prime}\sqrt{n};$$

or, in other words, if one calls $\lambda_{\prime}$ the mean of the n values of x^2, so that one has

$$\lambda_{\prime} = \frac{1}{n}s_{\prime},$$

the value of V given by formula (2) will express the probability that this mean $\lambda_{\prime}$ will be comprehended between the limits

$$k_{\prime} \pm \frac{2\alpha g_{\prime}}{\sqrt{n}};$$

and if one takes $\alpha = 1.8$, for example, there will be odds one hundred against one that the difference $\lambda_{\prime} - k_{\prime}$ will fall between the limits $\pm\frac{3.6g_{\prime}}{\sqrt{n}}$. The quantity $g_{\prime}$ being independent of n, it follows that this number n will be able always to become rather great in order that one may be able to neglect this difference, and to take the mean $\lambda_{\prime}$ for the value of the unknown $k_{\prime}$, one concludes from it

$$k_{\prime} = \int_0^\infty x^2 fx\,dx + \int_0^\infty f(-x)dx = \int_{-\infty}^\infty x^2 fx\,dx;$$

so that $k_{\prime}$ designates the same definite integral that the quantity l comprehends in the expression of g. One will have therefore also $l = \lambda$; and because of $k = \lambda$, there will result from it

$$g^2 = \frac{1}{2}(\lambda_{\prime} - \lambda^2),$$

in order to determine the value demanded of g.

By calling $s_{\prime}'$ the sum of the squares of the n values of y, given by the observation at the same time as those of x, and making

$$\lambda_{\prime}' = \frac{1}{n}s_{\prime}',$$

one will have also

$$g'^2 = \frac{1}{2}(\lambda'_, - \lambda'^2).$$

For a rifle known just, or for which one has found some values of λ and λ' null or negligible, one will have therefore simply

$$2g = \sqrt{2\lambda_,}, \qquad 2g' = \sqrt{2\lambda'_,}.$$

Consequently, between two rifles of this nature, the better under the relation of the horizontal deviations, will be the one for which the mean $\lambda_,$ of the squares of the distances to the axis of y will have the smallest value, and under the relation of the vertical deviations, it will be the rifle to which will correspond the smallest value of the mean $,\lambda'_,$ of the squares of the distances to the axis of x.

(6) Having measured the coordinates x and y of the points of the target where the center of the ball comes to strike at each shot, I suppose that one has calculated, not only the means λ, λ', $\lambda_,$, $\lambda'_,$, of these quantities and of their squares, but also the mean of their products, that is the sum of the positive or negative values of xy, divided by the number n of the tests, and I designate this mean by μ. If one has found, exactly or very nearly,

$$\lambda = 0, \quad \lambda' = 0, \quad \lambda_, - \lambda'_, = 0, \quad \mu = 0, \tag{3}$$

relative to the two straight lines that one has chosen for axes of x and of y, the same equations will hold equally with regard to two other rectangular axes passing, as the first, through the center C of the target.

In fact, let $x_,$ and $y_,$ be the coordinates of any point M reported to the two new axes, and θ the angle comprehended between the one of $x_,$ and the axis of x; one will have

$$\begin{aligned} x_, &= x\cos\theta - y\sin\theta, \\ y_, &= x\sin\theta + y\cos\theta; \end{aligned}$$

and if one calls β, β', $\beta_,$, $\beta'_,$, δ, the means of the n values of $x_,$, $y_,$, $x_,^2$, $\gamma_,^2$, $x_,y_,$, one will conclude from them

$$\begin{aligned} \beta &= \lambda\cos\theta - \lambda'\sin\theta, \\ \beta' &= \lambda\sin\theta + \lambda'\cos\theta, \\ \beta_, &= \lambda_,\cos^2\theta + \lambda'_,\sin^2\theta - 2\mu\sin\theta\cos\theta, \\ \beta'_, &= \lambda_,\sin^2\theta + \lambda'_,\cos^2\theta + 2\mu\sin\theta\cos\theta, \\ \delta &= (\lambda_, - \lambda_,^2)\cos\theta\sin\theta + \mu(\cos^2\theta - \sin^2\theta). \end{aligned}$$

Now, by virtue of equations (3), one deduces from these here:

$$\beta = 0, \quad \beta' = 0, \quad \beta_, - \beta'_, = 0, \quad \delta = 0;$$

that which there is concern to demonstrate.

One will have besides $\beta_, = \lambda_,$. It follows therefore that a rifle recognized just and equally good with respect to the two axes, the one horizontal and the other vertical, drawn through the point C, and for which the sum designated by μ will have been found equal to zero, will be also just and equally good with respect to every other straight line passing through this point. The mean distance from the points where the center of the ball will strike the target, to any straight line drawn through the point C, will approach more and more zero, in measure as the number n, that one supposes very great, will increase yet further, and formula (2) will express the probability that this mean will be comprehended between the limits $\pm\frac{\alpha\sqrt{2\lambda'}}{\sqrt{n}}$, which is the direction of the straight line to which it is reported.

(7). One is able further to compare the justness of the different arms, not under the relation of the deviations of the shots departing from a straight line passing through the center of the target, but relative to the deviations departing from this same center, and in every sense around this point.

For that, let r be the radius vector CM of any point M of the target, and θ the angle that this straight line makes with the axis of x; one will have

$$x = r\cos\theta, \quad y = r\sin\theta;$$

and all the possible values of x and y, positives and negatives, will result from the values of θ from $\theta = 0$ to $\theta = 2\pi$, and of those of r from $r = 0$ to $r = \infty$; so that this variable r must always be regarded as a positive quantity.

The probability that in firing a shot, the center of the ball will come to fall at a point of the circle described from point C as center and of a radius r, will be evidently a function of r; I will designate it by ψr; and by making $\frac{d\psi r}{dr} = \psi r$, the product $\psi r\,dr$ will be the infinitely small probability that the center of the ball will attain a point on the circumference of this circle. The function ψr will not be known to us; for of the increasing values of r, it will decrease so much more rapidly as the rifle will be better, and as the soldier will be more skilled, by supposing that he aims the best that he is able toward the center of the target. For one same arm and one same soldier, it will depend also on the distance of the soldier to the target; and its numerical value will increase generally, for each given value of r, when this distance will diminish.

All the possible values of r being comprehended from $r = 0$ to $r = \infty$, the sum of their probabilities is certitude, and one must have, consequently,

$$\int_0^\infty \psi r\,dr = 1.$$

There results from it that if one constructs the curve of which r and ψr are the abscissa and the ordinate of any point, the entire area of this curve will be unity, and the abscissa of its center of gravity will be the integral $\int_0^\infty r\psi r\,dr$ which will be presented just now.

This granted, we designate by σ the sum of the values of r which correspond to the center of the ball in any number n of tests; the probability that this sum will fall between the given limits $b \pm c$, will have for expression formula (1), in which one will make

$$\begin{aligned} \rho\cos\phi &= \int_0^\infty \psi r\cos zr\,dr, \\ \rho\sin\phi &= \int_0^\infty \psi r\sin zr\,dr, \end{aligned}$$

be extending, as one sees, the integrals to all the possible values of r, just as in n° 3, they extend to all those of x and of y.

If one makes also

$$\int_0^\infty r\psi r\,dr = \gamma_{\prime}, \quad \int_0^\infty r^2\psi r\,dr = \epsilon, \quad \frac{1}{2}(1-\gamma^2) = \omega^2,$$

and if one supposes that n is a very great number, one will transform formula (1) into formula (2) which will express the probability that the sum σ will be comprehended between the limits

$$n\gamma \pm 2\alpha\omega\sqrt{n};$$

or else, by making

$$\zeta = \frac{\sigma}{n},$$

in a manner that ζ is the mean of the n values of r, it will be also formula (2) which will express the probability that this mean will fall between the limits

$$\gamma \pm \frac{2\alpha\omega}{\sqrt{n}}.$$

(8). By giving to α a value great enough in order that the value of V given by formula (2), differs very little from unity, and by observing that the quantity ω is independent of the number n of the tests, one will be able therefore to regard as certain that the mean distance ζ to the point C, from the points where the center of the ball will attain the target, will converge indefinitely toward the quantity γ, in measure as the number n will increase more and more, and that for a given value of n, the limits of the difference $\zeta - \gamma$ will be so much tighter as the quantity ω will be smaller.

Reciprocally, one will be able to take for the approximate and very probable value of the constant γ, the mean ζ which will result from a very great number of shots. By a calculation similar to the one of n° 5, one will find also that the approximate and very probable value of ω^2 is $\frac{1}{2}(h-\zeta^2)$, by designating, in this same number n of shots, by h the mean of the squares of the distances to the center C, of the n points where the center of the ball will attain the target. The value of h will be deduced from those of the means that one has designated above by $\lambda_{\prime}$ and $\lambda'_{\prime}$; one will have evidently

$$h = \lambda' + \lambda'_{\prime}.$$

Even though the mean distances λ and λ' to the horizontal and vertical axes which are cut at the center C, both converge toward zero, and that one has found, consequently, for a very great number n of shots, of the values of λ and λ' very nearly null, the mean distance ζ to this point C tends not therefore nevertheless toward zero; it converges, to the contrary, toward a great constant γ which is able to be more or less considerable; and it is able even to happen that the mean distance to the center C, from the points where the center of the ball has attained the target, is greater than that of the same points to another point C' of the target. But in this case of $\lambda = 0$ and $\lambda' = 0$, the mean h of the squares of the distances of these same points to the center C, is always less than the mean of the squares of their distances to every other point C'.

In fact, let p and q be the coordinates of the point C' reported to the axes of x and y passing through the center C; by calling r' the distance $C'M$, one will have

$$r'^2 = (x - p)^2 + (y - q)^2 = r^2 - 2px - 2qy + p^2 + q^2.$$

If therefore one designates by h' the mean of the values of r'^2, and by observing that λ, λ', h, are the means of the values of x, y, r^2, there will result from it

$$h' = h - 2p\lambda - 2q\lambda' + p^2 + q^2.$$

One will determine the point C' for which this mean h' is a *minimum*, by equating to zero the differentials of h' with respect to p and to q, that which gives

$$p = \lambda, \qquad q = \lambda';$$

Consequently, the center C will be that point in the case of $\lambda = 0$ and $\lambda' = 0$.

If the constants γ and ω were known *a priori*, and if one described with point C as center, three circumferences of which the radii were $\gamma - \frac{2\alpha\omega}{\sqrt{n}}$, γ, $\gamma + \frac{2\alpha\omega}{\sqrt{n}}$, there would be the probability V expressed by formula (2), that is by taking, for example, $\alpha = 1.8$, odds around one hundred against one that the mean ζ of the distances to the point C, would come to end in the circular zone comprehended between the two extreme circumferences, and so much closer to the intermediate circumference, as the square root of n would be a greater number. Under the relation of the deviations around the center of the target, the goodness of a rifle depends therefore of two elements: on the one hand, it is so much better as the radius γ of this last circumference is smaller; and, on the other hand, it is preferable when the zone comprehended between the two other circumferences is tighter, for one same number n and one same value of α; or, in other words, it is also so much better as the constant ω has a smaller value. These two different quantities are not able to be, both, to the advantage of one same arm. It is possible that the constant γ is smaller for one rifle than for another, and that at the same time, the constant ω is less for the second rifle than for the first. But, in every case, if one takes for γ and ω their approximate values ζ and $\sqrt{\frac{1}{2}(h - \zeta^2)}$, the radius of the exterior circumference will be

$$\zeta + \frac{\alpha\sqrt{2(h - \zeta^2)}}{\sqrt{n}};$$

and one must always prefer the rifle for which this radius will be found smallest, for a given number of trials and for a value of α also given.

(9). By resuming all that which has been said in this memoir, one sees that under the different relations where the arms are able to be compared according to the tests of target shooting, their relative goodness depends on the mean of the deviations of a great number of shots, and on the probability of the limits between which this mean will be comprehended. If one gives, in all these comparisons, one same value to the quantity that I have designated by α, the probability of which there is concern will also be the same, and the relative goodness of the arms will depend no longer but on the magnitude of the mean deviations, or on their limits. Now, these limits, in the case of a very great number of tests, depend no longer on the unknown law of probability of the deviations at each shot; they depend only on certain definite integrals, dependents themselves on this law, but of which the approximate values are able to be deduced from the same result of the tests.

Thus, by conserving all the preceding notations, the limits of the mean deviation, on one side or the other of the vertical axis drawn through the center of the target, are

$$k \pm \frac{\alpha}{\sqrt{n}}\sqrt{2(l-k^2)}, \qquad (a)$$

in a number n of tests, supposed very great; those of the mean deviation, on one side or the other of the horizontal axis drawn through the same point, are at the same time,

$$k' \pm \frac{\alpha}{\sqrt{n}}\sqrt{2(l'-k'^2)}, \qquad (b)$$

and finally, those of the mean deviation about the point C, have for expression

$$\gamma \pm \frac{\alpha}{\sqrt{n}}\sqrt{2(\epsilon-\gamma^2)}, \qquad (c)$$

(10). Although the exact values of k, l, k', l', γ, ϵ, are able to be calculated *a priori*, only by making some purely gratuitous assumptions on the laws of probability of the deviations, or on the forms of the functions designated previously by fx, $f'y$, ψr, it will not be useless, however, to give an example of this calculation. Let therefore a and b be two lines of given magnitude, and we suppose that one has

$$\int fx = \frac{1}{a\sqrt{\pi}}e^{-\frac{x^2}{a^2}}, \qquad \int f'y = \frac{1}{b\sqrt{\pi}}e^{-\frac{y^2}{b^2}};$$

that which satisfies the conditions (n° 2)

$$\int_{-\infty}^{\infty} fx\,dx = 1, \qquad \int_{-\infty}^{\infty} f'y\,dy = 1.$$

According to the definite integrals that k, k', l, l', represent, one will have

$$k = 0, \quad k' = 0, \quad l = \frac{1}{2}a^2, \quad l' = \frac{1}{2}b^2;$$

the mean deviations, to which the limits (a) and (b) correspond, will converge therefore continually toward zero, in measure as the number n will increase more and more; and these limits will be $\pm\frac{\alpha a}{\sqrt{n}}$ in the horizontal sense, and $\pm\frac{\alpha b}{\sqrt{n}}$ in the vertical sense. By taking $\alpha = 1.8$, and supposing that a and b are a decimeter, there will be, by virtue of formula (2), odds of one hundred against one, for each of these mean deviations, that it will not be elevated by more than 0.018^{m} in one hundred tests.

The form of the function ψr will be deduced always from that which one will have supposed to each of the functions fx and $f'y$. In fact, the probability that the center of the ball will fall on the differential element $dxdy$ of the surface of the target is $fxf'ydxdy$ (n° 2); the probability ψr that the circle described with point C as center and with radius r will be therefore the integral of $fxf'y\,dxdy$ extended to all the points of this circle; but by replacing the orthogonal coordinates x and y by the polar coordinates r and θ, that is by setting $r\cos\theta$ and $r\sin\theta$ in place of x and y, it will be necessary also to replace the differential element $dxdy$ by $rdrd\theta$; one will have therefore

$$\psi r = \iint f(r\cos\theta)f'(r\sin\theta)r\,drd\theta;$$

and the integral relative to θ must be extended from $\theta = 0$ to $\theta = 2\pi$, and that which corresponds to r must be null when this variable will be zero. Consequently, if one differentiates this equation with respect to r, and if one sets ψr instead of $\frac{d\psi r}{dr}$, there will result from it

$$\psi r = \int_0^{2\pi} f(r\cos]\theta)f'(r\sin\theta)rd\theta.$$

In the example which we have chosen, we will have then

$$\psi r = \frac{1}{\pi ab}\int_0^{2\pi} e^{-\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)r^2} rdrd\theta.$$

One will verify without difficulty that the integral $\int_0^\infty \psi rdr$ is unity; for by interchanging the order of integration relative to r and θ, one will have first

$$\int_)^\infty \psi r = \frac{1}{\pi ab}\int_0^{2\pi}\left[\int_0^\infty e^{-\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)r^2} rdr\right]d\theta;$$

by effecting the integration relative to r, there will result

$$\int_0^\infty \psi rdr = \frac{ab}{2\pi}\int_0^{2\pi}\frac{d\theta}{b^2\cos^2\theta + a^2\sin^2\theta};$$

and by the ordinary rules, one will find

$$\int_0^\infty \psi rdr = 1.$$

In order to facilitate the integrations whence depend the values of γ and ϵ, I change r into another variable z, such that one has

$$\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)r^2 = z^2,$$

and, consequently,

$$\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)rdr = zdz.$$

The limits relative to z will be also $z = 0$ and $z = \infty$, and one will have

$$\gamma {=} \int_0^\infty r\psi rdr \ = \frac{a^2b^2}{\pi}\int_0^\infty e^{-z^2}z^2dz\int_0^{2\pi}\frac{d\theta}{(b^2\cos 2\theta + a^2\sin 2\theta)^{\frac{3}{2}}},$$
$$\epsilon = \int_0^\infty r^2\psi rdr {=} \frac{a^2b^2}{\pi}\int_0^\infty e^{-z^2}z^2dz\int_0^{2\pi}\frac{d\theta}{(b^2\cos 2\theta + a^2\sin 2\theta)^2}.$$

One has besides

$$\int_0^\infty e^{-z^2} z^2\,dz = \frac{1}{4}\sqrt{\pi}, \quad \int_0^\infty e^{-z^2} z^2\,dz = \frac{1}{2};$$

in the value of ϵ, the integral relative to θ will be obtained by the ordinary rules, whatever be the constants a and b; in the value of γ, it will be reduced to elliptic functions, when these constants will be unequal; by supposing $b = a$, for more simplicity, the integrations relative to θ will be effected immediately; one will obtain finally

$$\gamma = \frac{a}{4}\sqrt{\frac{1}{2}\pi}, \quad \epsilon = \frac{a^2}{4^2};$$

and the limits (c) will become

$$\frac{a}{4}\sqrt{\frac{1}{2}\pi} \pm \frac{\alpha a}{\sqrt{2n}}\sqrt{1 - \frac{1}{8}\pi}.$$

In measure as the number n of shots will increase more and more, the mean distance to the center of the target will approach therefore to be equal to the constant $\frac{a}{4}\sqrt{\frac{1}{2}\pi}$, of which the value would be around 0.3^{m}, by taking a decimeter for a. The difference between this mean distance and $\frac{a}{4}\sqrt{\frac{1}{2}\pi}$, will fall between the limits $\pm\frac{\alpha a}{\sqrt{2n}}\sqrt{1 - \frac{1}{8}\pi}$, tighter than those of the mean horizontal or vertical deviation, in the ratio of $\sqrt{1 - \frac{1}{8}\pi}$ to $\sqrt{2}$.

(11). The calculation of the values of k, k', l, l', γ, ϵ, according to one hypothesis on the form of the functions fx and $f'y$, is able to be only one example of calculation, proper to clarify the matter. In practice, it will be necessary always to take for these quantities their approximate values, deduced from the measure of the deviations.

We suppose therefore that when one has made a very great number n of tests, in which one has employed different soldiers, aiming at their best toward the center of the target, with one same rifle and at one same distance from the goal. We suppose also that these tests have given

$$\begin{array}{lll} \frac{1}{n}\sum x = \lambda, & \frac{1}{n}\sum y = \lambda', & \frac{1}{n}\sum r = \zeta, \\ \frac{1}{n}\sum x^2 = \lambda_{\prime}, & \frac{1}{n}\sum y^2 = \lambda'_{\prime}, & \frac{1}{n}\sum r^2 = h = \lambda_{\prime} + \lambda'_{\prime}; \end{array}$$

the sums Σ extending to all the points where the center of the ball has attained the target, of which the dimensions are great enough in order that all the shots are carried, that which is a necessary condition and always possible to fulfill. The approximate values of the six unknown constants will be

$$k = \lambda, \quad k' = \lambda', \quad l = \lambda_{\prime}, \quad l' = \lambda'_{\prime}, \quad \gamma = \zeta, \quad \epsilon = h.$$

This being, when one will make a new series of tests, which will be composed of a great number n' of shots fired with the same rifle, at the same distance from the target, and by the same soldiers, or by some others to which one supposes the same skill, there will be a probability V given by formula (2), for each of the mean distances to the center of the target and to the axes passing through this point, that it will be comprehended between the following limits.

For the vertical axis, the limits of the mean distance, will be

$$\lambda \mp \frac{\alpha}{\sqrt{n'}}\sqrt{2(\lambda - \lambda'^2)}; \qquad (a')$$

for the horizontal axis, they will be likewise

$$\lambda' \mp \frac{\alpha}{\sqrt{n'}}\sqrt{2(\lambda_{\prime} - \lambda_{\prime}'^2)}; \qquad (b')$$

finally the limits of the mean deviation around the point C, will have for expression

$$\zeta \mp \frac{\alpha}{\sqrt{n'}}\sqrt{2(h - \zeta^2)}; \qquad (c')$$

By taking $\alpha = 1.8$, for example, there will be very nearly odds of one hundred against one, for each of these three mean deviations, that it will fall between the limits which correspond to it.

Whatever approximations that are the values of the unknowns k, k', l, l', γ, ϵ, that we have employed, they are only probables and not certain; it would be able to result from them some doubt on the formulas (a'), (b'), (c'), and on their probability V; but the usage of the quantities λ, λ', $\lambda_{\prime}$, $\lambda_{\prime}'$, ζ, h, in the place of the six unknowns, is completely justified by that which has been demonstrated rigorously in one of the memoirs cited at the beginning of this here.[3]

The formulas (a'), (b'), (c'), joined to equation (2), contains all that which the calculus of probabilities is able to furnish in order to compare the different arms, after the distances observed, from the points where the center of the ball comes to strike the target, to its center, and to the horizontal and vertical axes passing through this point C.

[3] Additions to the *Connaissance des Tems* for the year 1832, pages 13 and following.