CALCULATION OF CHANCES

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Thirteenth Problem[1]

One demands what is the advantage of the banker in the game called "Pharaon."

The principal conditions are the following. The banker has a complete deck of 52 cards. When one has shuffled them, he takes them one by one, places them first one to his right, then one to his left, then anew one to his right, one to his left, etc. Each time that two cards have been thus placed the players are free to set a certain sum on one or many cards. When the card of the adversary of the banker presents itself at right (when it occupies therefore an odd rank) the banker wins the stake of the player, but the banker loses the same sum when the card presents itself at left (when it occupies an even rank). The banker takes the half of the stake, when the cards of the players follow themselves, namely first one of odd rank then one of even rank, excepting only the following case: if the adversaries of the banker have set on two cards and if these two are the last which he distributes, the banker wins all the stakes. The last card finally is not in favor of a person: it does not count.

We put $p =$ the number of cards which the banker disposes, and q the number of times that the card indicated by the player is found among the cards of the banker.

We act first as if the second condition did not exist; we suppose therefore that the banker wins immediately all the stake when one of the cards of his adversaries presents itself at right. The case is identical then to the one where the players have some white tickets and some black tickets, where each of them draw a ticket blindly, the banker first, next his adversary and where the one there wins the stake who obtains first a white ticket. According to the eighth Problem the numerator of the part of the banker will be therefore

$$\overbrace{\frac{\overline{p-1}\times\overline{p-2}\times\overline{p-3}}{1\times2\times3}\text{ etc.}}^{q-1\text{ terms}} + \overbrace{\frac{\overline{p-3}\times\overline{p-4}\times\overline{p-5}}{1\times2\times3}\text{ etc.}}^{q-1\text{ terms}} + \overbrace{\frac{\overline{p-5}\times\overline{p-6}\times\overline{p-7}}{1\times2\times3}\text{ etc.}}^{q-1\text{ terms}} + \text{etc.}$$

until the last term which is null. And the numerator of the part of his adversary is then

$$\overbrace{\frac{\overline{p-2}\times\overline{p-3}\times\overline{p-4}}{1\times2\times3}\text{ etc.}}^{q-1\text{ terms}} + \overbrace{\frac{\overline{p-4}\times\overline{p-5}\times\overline{p-6}}{1\times2\times3}\text{ etc.}}^{q-1\text{ terms}} + \overbrace{\frac{\overline{p-6}\times\overline{p-7}\times\overline{p-8}}{1\times2\times3}\text{ etc.}}^{q-1\text{ terms}}$$

Date: 1716.

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[1]Calculation of chances by means of algebra. pp. 104–107 of the edition of Vollgraff.

By subtracting this expression from the preceding, one finds the remainder

$$\overbrace{\frac{\overline{p-2}\times\overline{p-3}\times\overline{p-4}}{1\times 2\times 3}\text{ etc.}}^{q-2\text{ terms}} + \overbrace{\frac{\overline{p-4}\times\overline{p-5}\times\overline{p-6}}{1\times 2\times 3}\text{ etc.}}^{q-2\text{ terms}} + \overbrace{\frac{\overline{p-6}\times\overline{p-7}\times\overline{p-8}}{1\times 2\times 3}\text{ etc.}}^{q-2\text{ terms}} + \text{etc.}$$

It is there the numerator of the expression which represents the advantage of the banker. It is necessary to continue this series to that which the first number of the numerator becomes $q-2$: when it is smaller the fraction is null. The denominator must be $\frac{p\times\overline{p-1}\times\overline{p-2}\times\overline{p-3}}{1\times 2\times 3\times 4}$ etc. (q terms). But by the second condition this advantage is diminished exactly by half, because the advantage of the banker consists uniquely in his first turn and the banker obtains only the half of the stake of his adversary. As for the other chances, those to win or to lose the entire stake, they are precisely equal when p is even and when there is no first turn, seeing that each chooses at his turn.

If in the preceding formula $q = 1$, each expression is null; but because of the condition concerning the last card the advantage of the banker is $\frac{1}{p}$, if $q = 1$; then each term $= 1$; solely because of the before-last condition the last term must be 2.

The preceding formula differs from that of the author of the "Analyse." It differs equally from that given by Mr. Nicolas Bernoulli for the case where q is greater than 2

$$\frac{1}{4}\times\frac{q}{p-q+1}-\frac{\overline{q}.\overline{q-1}}{\overline{p-q+1}.\overline{p-q+2}}\times\frac{1}{8}+\frac{1}{16}\times \frac{\overline{q}.\overline{q-1}.\overline{q-2}}{\overline{p-q+1}.\overline{p-q+2}.\overline{p-q+3}}-\text{etc. } (q-1\text{ terms}).$$

Our formula has a greater number of terms, but we operate with whole numbers, and in that formula there one operates with some fractions. One can draw that formula from the preceding by adding the different expressions; only it is necessary each time to skip one.

First example.

Let $p = 34$ and $q = 3$, while 1 represents the sum staked on each card. The advantage of the banker is the $\frac{1}{44}$ and, if $q = 4$, $\frac{21}{682}$.

But if one notices that in this game the advantage of the banker consists uniquely in the fact that he wins the half of the stake when two cards of the player follow themselves in order as we have mentioned, one can reason as follows:

Second Example.

Suppose that someone has some letters A, B, C, D etc. to p, that one takes some of them, for example A, B, C etc. to q, and that one demands then in how many ways 2 of the chosen letters can occupy the first and the second place (let O be this number), in how many ways one of the q letters can occupy the third and another the fourth place, if they do not occupy the first and the second place (let P be this number), etc. Q represents the number which corresponds to the fifth and to the sixth place, and thus in sequence. One will find, by putting $p - q = r$,

$\overline{q}\times\overline{q-1}\times 1\times 2\times 3\times 4$ etc. to $p-2 = O$

$\overline{q}\times\overline{q-1}\times\overline{r}\times\overline{r-1}\times 1\times 2\times 3\times 4$ etc. to $p-4 = P$

$\overline{q} \times \overline{q-1} \times \overline{r} \times \overline{r-1} \times \overline{r-2} \times \overline{r-3} \times 1 \times 2 \times 3 \times 4$ etc. to $p-6 = Q$
$\overline{q} \times \overline{q-1} \times \overline{r} \times \overline{r-1} \times \overline{r-2} \times \overline{r-3} \times \overline{r-4} \times \overline{r-5} \times 1 \times 2 \times 3 \times 4$ etc. to $p-8 = R$, etc.

The total number of possible cases is $1 \times 2 \times 3 \times 4 \times 5$ etc. to p.

One finds thus for the advantage of the banker

$$1 + \frac{\overline{r}.\overline{r-1}}{\overline{p-2}.\overline{p-3}} + \frac{\overline{r}.\overline{r-1}.\overline{r-2}.\overline{r-3}}{\overline{p-2}.\overline{p-3}.\overline{p-4}.\overline{p-5}} + \frac{\overline{r}.\overline{r-1}.\overline{r-2}.\overline{r-3}.\overline{r-4}.\overline{r-5}}{\overline{p-2}.\overline{p-3}.\overline{p-4}.\overline{p-5}.\overline{p-6}.\overline{p-7}} + \text{etc.}$$

the whole multiplied by $\frac{\overline{q}.\overline{q-1}}{\overline{p}.\overline{p-1}}$.

If the adversary of the banker sets 1, all must still be multiplied by $\frac{1}{2}$. This formula has been found in the "Analyse" by the reduction of certain equations. We are arrived here to the same result by aid of combinations, this which is much more easy.

Third Example.

In the same manner one finds for the part of the banker in the game named "Bassette," when q is greater than 1,

$$-\frac{1}{3}\frac{q.\overline{p-q}}{p.\overline{p-1}} + \frac{1}{2} \times \frac{q}{p} - \frac{1}{4} \times \frac{\overline{q}.\overline{q-1}}{\overline{p}.\overline{p-q+1}} + \frac{1}{2} \times \frac{\overline{q}.\overline{q-1}.\overline{q-2}}{\overline{p}.\overline{p-q+1}.\overline{p-q+2}} + \text{ etc. } (q \text{ terms})$$

If $q = 1$, it is necessary still to add $\frac{1}{p}$.