CALCULATION OF CHANCES

NICOLAS STRUYCK

Twelfth Problem[1]

One demands what will be the advantage of the banker in the game named "Treize," if one decides that the game will end when the banker will have won or lost a single time.

The banker takes a complete deck of cards which is shuffled and cut. He says "ace" and turns a card. If this card is an ace, the banker wins the stake of the players, otherwise he says "two" and turns anew a card. If this card is a two, the banker wins, otherwise he says "three" and thus in sequence to 13. The thirteenth card must be a king. If the banker turns no cards that he announces, he must pay to his adversaries a sum equal to their stake, and his neighbor on the right obtains the bank; but if he announces a card which corresponds with that which he turns, he wins all the stakes and continues to say: ace, 2, 3, etc. as above. When finally there remain no more cards, the banker takes all the cards and shuffles them anew. He recommences then there where he had remained, until he wins or loses. If he wins, he recommences as we have said, and thus in sequence.

We suppose that one has n cards and that the colors, heart, diamond etc., are in number p. One has therefore p aces, p kings, etc. The total number of different ways in which one can take n cards is $1 \times 2 \times 3 \times 4 \times$ etc. to n. The total number of possible arrangements where one ace occupies the first place is $p \times 1 \times 2 \times 3 \times 4 \times$ etc. to $n-1$. If one calls 1 the stake that one can win by the game, the banker has therefore right to the part $\frac{p}{n}$. The total number of possible arrangements where 2 occupies the second place, if the ace does not occupy the first, is $p \times 1 \times 2 \times 3 \times 4 \times$ etc. to $n-1$, $-p^2 \times 1 \times 2 \times 3 \times 4 \times$ etc. to $n-2$. The banker has right to the part $\frac{p}{n} - \frac{p^2}{n \times \overline{n-1}}$ of the stake. The total number of cases, where a 3 will occupy the third place, when the first is not occupied by an ace nor the second by a 2 is $p \times 1 \times 2 \times 3 \times 4 \times$ etc. to $\overline{n-1}$, $-2p^2 \times 1 \times 2 \times 3 \times 4 \times$ etc. to $\overline{n-2}$, $+p^3 \times 1 \times 2 \times 3 \times 4 \times$ etc. to $\overline{n-3}$, so that the banker has right thence to $\frac{p}{n} - \frac{2p^2}{n \times \overline{n-1}} + \frac{p^3}{n \times \overline{n-1} \times \overline{n-2}}$, and in continuing to reason in the manner, one will find the expressions which follow, where one must know that the ordinary numbers, figuring in the numerators and which are not exponents, represent the offspring.

Date: 1716.

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[1]Calculation of chances by means of algebra. pp. 102–104 of the edition of Vollgraff.

For the ace	$\frac{p}{n}$
For the two	$\frac{p}{n} - \frac{p^2}{n.\overline{n-1}}$
For the three	$\frac{p}{n} - \frac{2p^2}{n.\overline{n-1}} + \frac{p^3}{n.\overline{n-1}.\overline{n-2}}$
For the four	$\frac{p}{n} - \frac{3p^2}{n.\overline{n-1}} + \frac{3p^3}{n.\overline{n-1}.\overline{n-2}} - \frac{p^4}{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}$
For the five	$\frac{p}{n} - \frac{4p^2}{n.\overline{n-1}} + \frac{6p^3}{n.\overline{n-1}.\overline{n-2}} - \frac{4p^4}{n.\overline{n-1}.\overline{n-2}.\overline{n-3}} + \frac{p^5}{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}}$

The first column has $\frac{n}{p}$ terms; the sum of the ordinary numbers which figure there is therefore $\frac{n}{p}$. In the second column there are $\frac{n}{p} - 1$ terms; the sum of the ordinary numbers there is $\frac{\frac{n}{p} \times \overline{\frac{n}{p}-1}}{1.2}$; one finds likewise for the third column $\frac{\frac{n}{p} \times \overline{\frac{n}{p}-1} \times \overline{\frac{n}{p}-2}}{1.2.3}$ and for the fourth $\frac{\frac{n}{p} \times \overline{\frac{n}{p}-1} \times \overline{\frac{n}{p}-2} \times \overline{\frac{n}{p}-3}}{1.2.3.4}$ and thus in sequence. The part of the banker is therefore expressed by the following formula

$$\frac{n}{p} \times \frac{n}{p} - \frac{\frac{n}{p} \times \overline{\frac{n}{p}-1}}{1.2} \times \frac{p^2}{n.\overline{n-1}} + \frac{\frac{n}{p} \times \overline{\frac{n}{p}-1} \times \overline{\frac{n}{p}-2}}{1.2.3} \times \frac{p^3}{n.\overline{n-1}.\overline{n-2}}$$
$$- \frac{\frac{n}{p} \times \overline{\frac{n}{p}-1} \times \overline{\frac{n}{p}-2} \times \overline{\frac{n}{p}-3}}{1.2.3.4} \times \frac{p^4}{n.\overline{n-1}.\overline{n-2}.\overline{n-3}} + \text{etc.}$$

or by reducing the fractions

$$1 - \frac{n-p}{1.2.\overline{n-1}} + \frac{\overline{n-p}.\overline{n-2p}}{1.2.3.\overline{n-1}.\overline{n-2}} - \frac{\overline{n-p}.\overline{n-2p}.\overline{n-3p}}{1.2.3.4.\overline{n-1}.\overline{n-2}.\overline{n-3}} + \text{etc.}$$

The number of terms must be equal to the one of units contained in $\frac{n}{p}$. We have therefore found here the same formula by which MR. NICOLAS BERNOULLI is served. But while the "Analyse" contains only the formula alone, we have added here the calculation which leads to it.

First Example

If $n = 52$ and $p = 4$, and if the total stake of the adversaries of which the banker is a, the advantage to the banker is $\frac{7672980411374433}{26816424180170625}a$.

Second Example

If $q = 1$, that is if all the cards are different, we have a deal in the particular case which MR. BERNOULLI resolved in this "Latin remarks." The advantage of the banker is then expressed by the following series which must have a number of terms equal to the number of cards:

$$1 - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \frac{1}{1.2.3.4.5} - \frac{1}{1.2.3.4.5.6} + \text{ etc.}$$