UYTREEKENING DER KANSEN IN HET SPEELEN, DOOR DE ARITHMETICA EN ALGEBRA, BENEEVENS EENE VERHANDELING VAN LOOTERIJEN EN INTEREST

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Excerpted from
Calculation of chances in games (1716).

TO THE READER

The Calculation of Chances has not been known to the ancients: Mister CHRISTIAN HUYGENS is nearly the first who has written on this subject. His treatise is from the year 1657: translated into Dutch it is found in the "Mathematical Exercises" of professor VAN SCHOOTEN. No notable progress has been accomplished in the calculus of chances before the year 1708. In that year we have printed at Paris an "Essai d'Analyse sur les jeux de Hasard," where the Author tries to continue these calculations, as we also on our side have endeavored to do it.

It is not necessary to prove the utility and the advantage that these calculations procure for us. We would not dare besides to attribute to them a value as high as that which M. JACQUES BERNOULLI quite wants to attribute to them: we ourselves prefer to hold to the words of which M. HUYGENS that we just named serves himself in his letter to professor VAN SCHOOTEN "if someone studies thoroughly a little these things, I quite wish to believe that he will find that he was not concerned with a simple game, but with the principles and foundations of an interesting and profound speculation. Also I believe that my problems touching this matter, will not be considered as easier than those of Diophantos, but perhaps more diverting, because they contain another thing than the simple properties of numbers."

It is true that some persons imagine that it is impossible to make a calculation of chances of a game and that all depends only on chance. But experience shows that with two ordinary dice one will cast more often seven than twelve points and the same thing is able to hold with the calculation of chances, seeing that there are six cases which give seven points and one alone which gives twelve. Some others go still further by superstition, and believe that good luck or bad luck depends upon the persons, on the time and on the place.

We serve ourselves with the following hypothesis which appears not unreasonable to us: the whole of the stakes must be divided in ratio of the chances that each has to win the game. Because if there are two players A and B, having both the same chance to win a certain game, what reason would one be able to invent which would carry us to believe that A will triumph rather than B? There is evidently none any longer to sustain the contrary. Consequently, if they do not wish that the result depend upon chance, in other terms if they wish to divide, it is clear that each must have the half.

Date: 1716.

See *Les Oeuvres de Nicolas Struyck* translated from the Dutch by J. A. Vollgraff (1912). Selections translated from the French by Richard J. Pulskamp, Department of Mathematics and Computer Science, Xavier University, Cincinnati, OH. Prepared this July 19, 2009.

We treat first of the calculation of chances by means of arithmetic, in starting from elementary principles. Indeed, we have developed and proved all that which would be able to hold in any difficulty; it suffices therefore to comprehend that of knowing the ordinary rules.

We have made likewise in the calculation of chances by means of algebra, leaving nothing unexplained if it is not that which is quite easy to comprehend, excepting perhaps the last Problem. We have in general omitted the development of the Examples, seeing that the method is always the same; for when one possesses the algebraic formula, it is easy to replace the letters by some numbers. As for the fifth and sixth Example of the second Problem, it is necessary to seek that which we say in the last page of the "Calculation of the lotteries and interests."

In the treatise on the lotteries and on the interests, one will note how much the algebra often shortens the calculations which without it are nearly impossible because of their tedious length.

5 Problems of Christiaan Huygens (pp. 32–45)

One sees clearly that one could deduce from the Problems which precede a crowd of Examples beyond those which we have named but because the solution of each of them is sufficiently evident by analogy with that of the preceding Examples, we will content ourselves to make here follow the solution of the **5 Problems** which Mr. Christian Huygens has previously proposed to the amateurs.

I. A and B play one against the other with 2 dice, with the following conditions. A will be winner if he casts 6 points, and B if he casts 7 of them. A will make the first coup. After him B will make two of them consecutively, next A will make two of them, and thus consecutively to that which one or the other will have won. One demands the ratio of the chances to win that A and B have respectively. Response: $10355 : 12276$.

A has 5 chances to win, B has 6 of them. One can make 36 different coups with 2 ordinary dice. We will calculate first the value of the first coup of A, and next that of the double coups which the players make turn by turn.

$$
\begin{array}{rl|rl}
\multicolumn{2}{c|}{\text{Chances}} & \multicolumn{2}{c}{\text{Chances}} \\
5 \;—\; 1 & \left\{\begin{array}{c} 5 \\ 0 \end{array}\right. & 6 \;—\; \frac{31}{36} & \left\{\begin{array}{c} \frac{31}{6} \\ 0 \end{array}\right. \\
31 \;—\; 0 & & 30 \;—\; 0 & \\
\hline
36 & 5 & 36 & \frac{31}{6} \\
 & 36— & & 36— \\
\text{The first coup of A} & \frac{5}{36} & \text{The first coup of B} & \frac{31}{216} \\
 & 1— & & \frac{31}{16} \\
\hline
\text{Remainder} & \frac{31}{36} & \text{Remainder} & \frac{155}{216}
\end{array}
$$

Chances

$$\begin{array}{rl} 6 & \text{—}\ \frac{155}{216} \\ 30 & \text{—}\ 0 \\ \hline 36 & \end{array} \qquad \left\{ \begin{array}{c} \frac{155}{36} \\ 0 \end{array} \right. \qquad \begin{array}{c} \hline \frac{155}{36} \\ 36\text{—} \end{array}$$

$$\begin{array}{rc} \text{The second coup of B} & \frac{155}{1296} \\ & \frac{155}{216} \\ \text{Remainder} & \frac{775}{1296} \end{array}$$

Chances

$$\begin{array}{rl} 5 & \text{—}\ \frac{775}{1296} \\ 31 & \text{—}\ 0 \\ \hline 36 & \end{array} \qquad \left\{ \begin{array}{c} \frac{3875}{1296} \\ 0 \end{array} \right. \qquad \begin{array}{c} \hline \frac{3875}{1296} \\ 36\text{—} \end{array}$$

$$\begin{array}{rc} \text{The first coup of A} & \frac{3875}{46656} \\ & \frac{775}{1296} \\ \hline \text{Remainder} & \frac{24025}{46656} \end{array}$$

Chances

$$\begin{array}{rl} 5 & \text{—}\ \frac{24025}{46656} \\ 31 & \text{—}\ 0 \\ \hline 36 & \end{array} \qquad \left\{ \begin{array}{c} \frac{120125}{46665} \\ 0 \end{array} \right. \qquad \begin{array}{c} \hline \frac{120125}{46656} \\ 36\text{—} \end{array}$$

$$\begin{array}{rc} \text{The second coup of A} & \frac{120125}{1679616} \\ \text{The first coup of A} & \frac{3875}{46656} \\ \hline \text{For two coups of A} & \frac{259625}{1679616} \end{array}$$

$$\begin{array}{lc} \text{The first coup of B} & \frac{31}{216} \\ \text{The second coup of B} & \frac{155}{1296} \\ \hline \text{The 2 coups of B} & \frac{341}{1296} \end{array}$$

And as the numbers which correspond to the 2 coups that B and A make one after the other conserve between them the same proportion to infinity, and which is therefore equal to the proportion of the sums of the numbers corresponding to the first two coups which the players make one after the other, it is necessary to divide $\frac{31}{36}$, the part of the game which remains for the two players after the first coup of A, in ratio of these sums, that which is made in the following manner. By dividing, in order to shorten the calculation, the numerators by 31 and the denominators by 1296, one sees that these sums are the one to the other as 11 is to $\frac{8375}{1296}$. The sum of these numbers is $\frac{22631}{1296}$.

$$\frac{22631}{1296} \text{—B } 11 - \frac{31}{36} \left\{ \text{the part of B will be } \frac{12276}{22631} \right. .$$

If you subtract this number from unity, there remains $\frac{10355}{22631}$; this is the part of A. The chance to win of player A is therefore to that of B as 10355 is to 12276. That which it was necessary to find.

II. Three players A, B and C take 12 tickets, of which 4 white and 8 black. They play with the following conditions: The one from among them who will have first drawn a white will be winner. A will draw first, then B, next C, next A anew, and thus in order by turn. One demands the ratios of the chances of the three players.

One can understand this proposition in two ways. In first place, one can admit that each black ticket that one has drawn is returned and added to those which remain, so that each can draw a ticket from among all those which have been given. In second place one can admit that the black tickets which one has drawn are not returned. In this last case one sees at the ninth coup at the latest who wins the game, while in the first case the game can endure until infinity. First we will make the first supposition, next the second.

$$
\begin{array}{llllllllll}
\text{A} & 4 & \text{— } 1 & \left\{\begin{array}{l}4\\0\end{array}\right. & 4 & \text{— } \frac{2}{3} & \left\{\begin{array}{l}2\frac{2}{3}\\0\end{array}\right. & 4 & \text{— } \frac{4}{9} & \left\{\begin{array}{l}\frac{16}{9}\\0\end{array}\right. \\
 & 8 & \text{— } 0 & & 8 & \text{— } 0 & & 8 & \text{— } 0 & \\
\hline
 & 12 & & 4 & 12 & & 2\frac{2}{3} & 12 & & \frac{16}{9} \\
 & & & 12\text{—} & & & 12\text{—} & & & 12\text{—} \\
 & & & \text{A } \frac{1}{3} & & & \text{B } \frac{2}{9} & & & \text{C } \frac{4}{27} \\
 & & & 1\text{—} & & & \frac{2}{3} & & & \\
\hline
 & \text{Remainder} & & \frac{2}{3} & \text{Remainder} & & \frac{4}{9} & & &
\end{array}
$$

The chances of A, of B and of C will maintain among them the same proportion from the first 3 coups until infinity. It suffices therefore to take the chances already calculated and to give them the same denominator.

$$
\begin{array}{lllll}
\text{A } \frac{1}{3} & \text{—} & \text{B } \frac{2}{9} & \text{—} & \text{C } \frac{4}{27} \quad 27 \\
\hline
\text{A } 9 & \text{—} & \text{B } 6 & \text{—} & \text{C } 4
\end{array}
$$

It is to these numbers that their chances are proportional.

An analogous solution can be given of the following problem. A, B, C and D possess some pieces of money. It is the turn of A to cast them: all those which present heads are for him. B casts those which remain; he takes all those which present heads. C, then D, are the same. A casts anew all those which remain and takes those which present heads, B makes the same thing with those which remain then, and thus consecutively until they have all been won. What will be the chance of each player?

Seeing that each wins the half of the pieces of money that he casts, the parts of which the chances of the players are composed are the following.

$$
\begin{array}{lllll}
\text{At the first turn} & \text{A } \frac{1}{2}, & \text{B } \frac{1}{4}, & \text{C } \frac{1}{8}, & \text{D } \frac{1}{16}, \\
\text{At the second} & \text{A } \frac{1}{32}, & \text{B } \frac{1}{64}, & \text{C } \frac{1}{128}, & \text{D } \frac{1}{256}, \\
\text{At the third} & \text{A } \frac{1}{512}, & \text{etc.} & &
\end{array}
$$

As these parts conserve among them the same proportion until infinity, one sees clearly that in order to find the part of each, it suffices to divide unity into 4 parts, having among them the same ratios as the numbers $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, etc. This is done in the following manner:

$$
\begin{array}{c}
\frac{1}{2} \\ \frac{1}{4} \\ \frac{1}{8} \\ \frac{1}{16} \\ \frac{15}{16}
\end{array}
\qquad
\begin{array}{l}
\text{A } \frac{1}{2} \\ \text{B } \frac{1}{4} \\ \text{C } \frac{1}{8} \\ \text{D } \frac{1}{16}
\end{array}
\quad \text{—} \quad
\left\{\begin{array}{l}
\text{A } \frac{8}{15} \\ \text{B } \frac{4}{15} \\ \text{C } \frac{2}{15} \\ \text{D } \frac{1}{15}
\end{array}\right\}
\quad \text{These are the there numbers demanded.}
$$

But those who know algebra, could find the solution in the following manner. We suppose the number of players is s and the number r indicates the rank of the player of which one seeks the part, then the value of the first coup is for him $\frac{1}{2^r}$, the value of the second coup is $\frac{1}{2^{r+s}}$ and thus consecutively following a geometric progression to infinity. In order to find the sum, carry the first term $\frac{1}{2^r}$ to the square which gives $\frac{1}{2^{2r}}$. Divide this square by the difference of $\frac{1}{2^r}$ and of $\frac{1}{2^{r+s}}$, that is by $\frac{2^s-1}{2^{r+s}}$. There will come for the sum of the progression, of which for the part of any player $\frac{2^{s-r}}{2^{s-1}}$.

When one has $s = 4$ and $r = 2$, one finds for the part of the second player $\frac{4}{15}$. This same formula can be deduced from our formula which is found in the "Calculation of Chances by Algebra" a little before the first Example, belonging to the third Problem; because if one puts $a = 1$, $b = 1$ and $c = 2$, one obtains the preceding formula.

In order to resolve the Problem in the second hypothesis we operate as follows:

$$\begin{array}{rccc}
\text{A} & 4 & -1 & \left\{\begin{array}{c}4\\0\end{array}\right.\\
 & 8 & -0 & \\
\hline
 & 12 & & 4\\
 & & & 12-\\
 & & & \text{A}\ \frac{1}{3}\\
 & & & 1\\
\hline
\text{Remainder} & & & \frac{2}{3}
\end{array}
\qquad
\begin{array}{rccc}
 & 4 & -\frac{2}{3} & \left\{\begin{array}{c}2\frac{2}{3}\\0\end{array}\right.\\
 & 7 & -0 & \\
\hline
 & 11 & & 2\frac{2}{3}\\
 & & & 11-\\
 & & & \text{B}\ \frac{8}{33}\\
 & & & \frac{2}{3}\\
\hline
\text{Remainder} & & & \frac{14}{33}
\end{array}
\qquad
\begin{array}{ccc}
4 & -\frac{14}{33} & \left\{\begin{array}{c}1\frac{23}{33}\\0\end{array}\right.\\
6 & -0 & \\
\hline
10 & & 1\frac{23}{33}\\
 & & 10-\\
 & & \text{C}\ \frac{28}{165}
\end{array}$$

And in continuing in the same manner one will find that which follows:

$$\begin{array}{lcccccc}
\text{At the first turn} & \text{A} & \frac{1}{3}, & \text{B} & \frac{8}{33}, & \text{C} & \frac{28}{165},\\
\text{At the second turn} & & \frac{56}{495} & & \frac{7}{99} & & \frac{4}{99},\\
\text{At the third turn} & & \frac{2}{99} & & \frac{4}{495} & & \frac{1}{495},\\
 & \text{A} & \frac{7}{15}, & \text{B} & \frac{53}{165}, & \text{C} & \frac{7}{33},\\
\hline
 & & & & & & \quad 165\\
 & \text{A} & 77, & \text{B} & 53, & \text{C} & 35.
\end{array}$$

Here are the numbers which make known the proportions of the chances among them.

But we have resolved the same Problem in another fashion yet, by the theory of combinations. This solution is the following:

It is certain that the first coup has for A the value $\frac{1}{3}$. If one considers next in how many ways 12 objects can be taken 2 by 2, one will find the number 66. In order to find next the sum of the values of the first coup of A and of the first coup of B, it is necessary to calculate in how many ways they can draw two black tickets, that is in how many ways 8 black tickets can be taken 2 by 2. One finds that this can take place in 28 ways. Subtract this number from 66; there remains 38. They have therefore right to $\frac{38}{66}$ or $\frac{19}{33}$ of the stake. From this part A must have $\frac{1}{3}$; there remains therefore for B $\frac{8}{33}$. Seek likewise the total value of the coup of A, of B and of C, as follows:

$$
\begin{array}{llll}
2 & & & \\
\cancel{4} & & & \\
\cancel{1}\cancel{2} - 11 - 10 & 8 - 7 - \cancel{6} & 11 & \\
1 - \cancel{2} - \cancel{3} & 1 - \cancel{2} - \cancel{3} & \underline{\quad 2} & \\
 & & 22 & \\
 & & \underline{\quad 10} & \\
 & & 220 & \text{Total number of chances} \\
 & 8 \times 7 = & 56 & \text{Number of unfavorable chances.} \\
 & & \overline{\quad 164} & \text{Number of favorable chances.}
\end{array}
$$

$$
\begin{array}{rl}
\text{Then the value of the coup of A, of B and of C is } \frac{164}{220} \text{ or} & \frac{41}{55} \\
\text{That of the coup of A and of B is} & \underline{\frac{19}{33}} \\
\text{Remainder for C} & \frac{28}{165}.
\end{array}
$$

And in continuing in the same manner one will find the parts of each player as before.

One sees easily how one can by the aid of this proposition find by arithmetic the advantage of the banker in the game named “Pharaon”: the 52 cards constitute the total number of tickets, and all passes as if 2 persons only drew. The cards on which one hazards hold the place of the white tickets. The only difficulty of the problem consists in the pain that it is necessary to take in order to make the numerical calculation.

III. A wagers with B that of 40 cards, of which 10 of each color, he will draw 4 of them in such manner that he will have one of each color of them. One will find here that the chance of A is to that of B as 1000 is to 8139.

A can from the 10 hearts draw one card in 10 different ways; likewise from the diamonds, the clubs and the spades. This is why the number of chances favorable to player A is 10^4. Seek next in how many ways one can from 40 cards take 4 of them, without having ever the same.

$$
\begin{array}{lrcrl}
10 \quad 13 \quad 19 & & & & \\
\cancel{4}\cancel{0} - \cancel{3}\cancel{9} - \cancel{3}\cancel{8} - 37 & 37 & & 10 & \\
1 - \cancel{2} - \cancel{3} - \cancel{4} & \underline{19} & & \underline{10} & \\
 & 703 & \text{mult.} & 100 & \\
 & \underline{13} & & \underline{10} & \\
 & 9139 & & 1000 & \\
 & \underline{10} & & \underline{10} & \\
\text{Total number of chances} & 91390 & & 10000 & \text{number of chances} \\
 & 10000 & & & \text{favorable to A.}
\end{array}
$$

The chance of B is to that of A as 8139 is to 1000. One can arrive to the same result by calculating the part which returns to the player after he has drawn one or many cards. Suppose that the player draws the cards one after the other. It is necessary to consider that if A had already drawn three different cards, he must from the 37 cards remaining draw from them yet one different from the first three. There are then yet 10 of them which make A win, and 27 which make B win. The stake being 1, A has right to $\frac{10}{37}$. If A has already taken 2 cards, for example a diamond and a heart, there are in all yet 38 cards, among which 10 clubs and 10 spades. If he draws one of these last two colors, it will be in the preceding case, that is he will have right to $\frac{10}{37}$. There are therefore 20 chances to obtain $\frac{10}{37}$ and 18 chances to obtain nothing. His part is therefore then according to the solution of the first Problem $\frac{100}{703}$. If A had drawn a diamond the first time, there would remain yet 39 cards, among which there would be only 9 diamonds which could make him lose. He would have therefore 30 chances to obtain $\frac{100}{703}$ and 9 chances to obtain nothing. His part is then $\frac{1000}{9139}$ and as he cannot take a wrong card the first time, it is there also his part at the beginning of the game. That of B will be therefore $\frac{8139}{9139}$. The chances of the two players have between them the ratio $1000 : 8139$, this which accords with the result of the preceding calculation.

IV. Having taken 12 tickets, of which 4 white and 8 black, A wagers with B to draw at random 7 tickets of which 3 white: one demands the ratio of the chance of A to that of B.

Seek first in how many ways one can from 12 tickets take 7 of them, next in how many ways one can take the 8 black tickets 4 by 4 and the 4 white 3 by 3.

$\overset{3}{\cancel{12}} - 11 - \overset{2}{\cancel{10}} - 9 - 8 - \cancel{7} - \cancel{6}$	$\cancel{8} - 7 - \overset{2}{\cancel{6}} - 5$	$4 - \cancel{3} - \cancel{2}$
$\cancel{1} - \cancel{2} - \cancel{3} - \cancel{4} - \cancel{5} - \cancel{6} - \cancel{7}$	$1 - \cancel{2} - \cancel{3} - \cancel{4}$	$1 - \cancel{2} - \cancel{3}$
11	7	4
9	2	70
99	14	280 chances
8	5	to take 3
792 different ways to take 7 tickets	70	white and 4 black.

Subtract 280 from 792. There remains 512. The chance of A is then to that of B as 280 is to 512, or as 35 is to 64, under the hypothesis that A must have neither more nor less than 3 white among the 7 tickets. But as one can equally interpret the announcement of the problem in this sense that A would win his wager if among the 7 tickets he drew more than 3 white, as well as when he draws 3 of them, it is necessary to calculate in how many ways one can take 8 tickets 3 by 3: the 4 white tickets could be taken in one way alone.

$8 - 7 - \cancel{6}$		
$1 - \cancel{2} - \cancel{3}$		792
there comes	56	336
	280	456
	336	

The chance of A will be to that of B as 336 is to 456 or as 42 is to 57. It is there the result demanded.

V. Having taken 12 pieces of money each, A and B play with 3 dice with this condition that when one casts 11 points, A must give a piece of money to B, but when one casts 14 of them, B must give a piece of money to A, and that the one there will be winner who will be the first in possession of all the pieces of money. One must find that the chance of A is to that of B as 244146025 is to 282429536481.

With three dice one can make 27 coups which give 11 and 15 coups which give 14 points. The ratio of these numbers is equal to $9 : 5$. Suppose that when B has a piece of money, and A all the others, B had yet right to a part of the stake equal to 1 florin. The total value of his chances is then 14 florins.

Among these chances there are 5 which give to him zero, the 9 others which give to him 2 pieces of money are worth therefore together 14 florins. The value of each chance is therefore $1\frac{5}{9}$ fl. It is there the part of B when he has two pieces of money. In order to calculate, by starting there, what is his part when he has 3 pieces of money, we remark, as before, that there are in all 14 chances, having a mean value of $1\frac{5}{9}$ fl., this which makes in all $21\frac{7}{9}$ florins. Among these there are 5 chances for him to possesses a piece of money; his part is then 1 florin; this which makes 5 florins for the 5 chances. If we subtract this number from $21\frac{7}{9}$, there remains $16\frac{7}{9}$ for the 9 other chances, this which makes $1\frac{70}{81}$ fl. for each of them. It is there the part of B when there are 3 pieces of money. One finds in the same manner the part of B when he has 4 pieces of money, by the following calculation:

All the chances

$$\begin{array}{ccll} 14 & -1\frac{70}{81} & \left\{\begin{array}{r} \text{fl. } 26\frac{8}{81} \\ 7\frac{7}{9} \end{array}\right. & \\ \underline{\;\;5\;\;} & -1\frac{5}{9} & & \\ 9 & & \begin{array}{c} \overline{18\frac{26}{81}} \\ \text{divided by } 9 \end{array} & \left\{\begin{array}{l} \text{fl. } 2\frac{26}{729}\text{, when B has 4} \\ \text{pieces of money} \end{array}\right. \end{array}$$

By continuing in the same manner one must seek the part of B when he has 12 pieces of money, and next his part when he has 24 or even all the pieces of money. By taking the first of these numbers for numerator and the last for denominator of a fraction, and in simplifying this fraction, one finds the part sought from B.

1 piece	2 pieces	3 pieces	4 pieces
1fl.	$1\frac{5}{9}$ fl.	$1\frac{70}{81}$ fl.	$2\frac{26}{729}$
	1	$1\frac{5}{9}$	$1\frac{70}{81}$
	$\frac{5}{9}$	$\frac{25}{81}$	$\frac{125}{729}$

But if one notes that the part to which B has right increases each time that he wins a piece of money, in such manner that the numbers which represent these increases form a geometric progression, one sees that it suffices to take the sum of the terms of a geometric progression, of which the number of terms is equal to the one of the pieces of money of B, which begins with unity and of which the ratio is $\frac{5}{9}$. This sum is the numerator. In order to find the denominator, one takes the sum of a number of terms of this same progression equal to the sum of the numbers of pieces of money of A and of B. Now, in order to seek the sum of a certain number of terms of this progression, it is necessary to take the first term, that is 1, to subtract from it the term which follows the last, that is $\frac{5^{12}}{9^{12}}$ in the numerator and $\frac{5^{24}}{9^{24}}$ in the denominator, and to divide the rest by unity diminished by the ratio. One finds thus the sums demanded. But as the division must be effected in the numerator as well as in the denominator, one can omit it. The part which reverts to B is thus

$$\text{Numerator} \qquad 1 — \frac{5^{12}}{9^{12}}$$
$$\text{Denominator} \qquad 1 — \frac{5^{24}}{9^{24}}$$

Now, it is certain that the product of the sum and of the difference of two numbers is equal to the square of the greatest number diminished by the square of the smallest. Because in multiplying the sum of the two numbers by the greatest only, one would find for result the sum of the square of the greatest number and of the product of the two numbers; and by multiplying this sum by the smallest number, one finds the same result, but this time with the square of the smallest number. By subtracting these two products from one another, one obtains a remainder equal to the difference of the two squares. In the fraction considered the numerator is the difference of 2 numbers, and the denominator the difference of their squares. It follows that if one divides the denominator by the numerator, one finds unity in the numerator and the sum of the numbers in the denominator; the part of B is therefore

$$\frac{1}{1+\frac{5^{12}}{9^{12}}},$$

or else, if one renders the numerator and the denominator whole,

$$\frac{9^{12}}{9^{12}+5^{12}}.$$

The part of A is therefore

$$\frac{5^{12}}{9^{12}+5^{12}},$$

and the chance of A is to that of B as 244140625 is to 282429536481.

The theorem which we just demonstrated and which has served us to simplify the considered fraction has no need to be demonstrated for those who know algebra: we have given the demonstration for those who understand only ordinary arithmetic. The number before which is placed the $+$ must be added to the one which precedes; but the one before which is found the $-$ sign must be subtracted from the preceding number. When one writes 5^{12}, this wants to say that it is necessary twelve times to multiply the number 5 by itself: in this fashion 5^2 is equal to 25, 5^3 to 125, 5^4 to 625, etc.

We show by the following example that the same method can be employed when the number of pieces of money of A and of B are not the same.

Two persons A and B play with 2 ordinary dice. A has two pieces of money, B has 5 of them. A will give a piece to B when one casts 12 points, B will give one of them to A when one casts 11 points. One demands the chance to each player.

A has 2 chances to win one piece of money against B one chance. Suppose that the chance of A has yet the value of 1 florin, when he possesses only 1 piece of money.

There are in all 3 chances; they are worth 3 florins. Among them there is one chance which is worth 0, the two others are worth therefore together 3 florins. This makes $1\frac{1}{2}$ florin for each chance, this which is the part of A when he has 2 pieces of money. We will calculate now the part which is due to him when he has all the pieces: this part must be equal to the stake.

$$\begin{array}{ll}
\text{Chances} & \\
3 \;\; — 1\tfrac{1}{2} & \left\{\begin{array}{r} 4\tfrac{1}{2} \\ 1 \end{array}\right. \\
\underline{1 \;\; — 1} & \underline{\qquad\quad} \\
\;\;2 & 3\tfrac{1}{2} \\
 & 2 — \\
 & \text{A } 1\tfrac{3}{4},
\end{array}$$

when he has 3 pieces of money.

$$\begin{array}{ll}
\text{Chances} & \\
3 \;\; — 1\tfrac{3}{4} & \left\{\begin{array}{r} 5\tfrac{1}{4} \\ 1\tfrac{1}{2} \end{array}\right. \\
\underline{1 \;\; — 1\tfrac{1}{2}} & \underline{\qquad\quad} \\
\;\;2 & 3\tfrac{3}{4} \\
 & 2 — \\
 & \text{A } 1\tfrac{7}{8},
\end{array}$$

when he has 4 pieces of money.

$$\begin{array}{ll}
\text{Chances} & \\
3 \;\; — 1\tfrac{7}{8} & \left\{\begin{array}{r} 5\tfrac{5}{8} \\ 1\tfrac{3}{4} \end{array}\right. \\
\underline{1 \;\; — 1\tfrac{3}{4}} & \underline{\qquad\quad} \\
\;\;2 & 3\tfrac{7}{8} \\
 & 2 — \\
 & \text{A } 1\tfrac{15}{16},
\end{array}$$

when he has 5 pieces of money.

$$\begin{array}{ll}
\text{Chances} & \\
3 \;\; — 1\tfrac{15}{16} & \left\{\begin{array}{r} 5\tfrac{13}{16} \\ 1\tfrac{7}{8} \end{array}\right. \\
\underline{1 \;\; — 1\tfrac{7}{8}} & \underline{\qquad\quad} \\
\;\;2 & 3\tfrac{15}{16} \\
 & 2 — \\
 & \text{A } 1\tfrac{31}{32},
\end{array}$$

when he has 6 pieces of money.

$$\begin{array}{ll}
\text{Chances} & \\
3 \;\; — 1\tfrac{31}{32} & \left\{\begin{array}{r} 5\tfrac{29}{32} \\ 1\tfrac{15}{16} \end{array}\right. \\
\underline{1 \;\; — 1\tfrac{15}{16}} & \underline{\qquad\quad} \\
\;\;2 & 3\tfrac{31}{32} \\
 & 2 — \\
 & \text{A } 1\tfrac{63}{64},
\end{array}$$

when he has 7 pieces of money.

One sees then: the stake being $1\frac{63}{64}$, A has right to $1\frac{1}{2}$; when the stake is 1, one finds therefore for the part of A $\frac{96}{127}$ and for that of B $\frac{31}{127}$. Their chances are therefore between them in the ratio $96 : 31$, that is A can wager a little more than 3 against 1 that he will win.

If in the preceding example A must give 1 piece of money to B when one casts 8, and B 1 piece of money to A when one casts 6 points, the chances to win a piece of money would be equal, therefore in the ratio $1 : 1$. If then the part of A is equal to 1 florin when he posses no more than one piece of money, his part would be of 2 florins when he has 2 pieces, of 3 florins when he has 3, etc. The chances of the players have then between them a ratio equal to the one of the number of pieces of money that each of the two possesses.

In this case the chance of A is therefore to that of B as 2 is to 5. But when there are more than 2 players the problem is more arduous. We will make follow here an example where there are three players.

There are 3 players A, B and C who play with 2 ordinary dice. A has 3 pieces of money, B has 1 of them, C has 2 of them. A will receive one piece from B and one from C when one casts 7 points. B wins one piece from A and one from C when one casts 6 with the same dice, and C one piece from A and one from B, when one casts 5. The one who has no more pieces, exits from the game and the 2 others continue to play. The one who obtains first all 6 pieces of money, is the winner. One demands the ratio of the chances of the three players among them.

Response. The chances of A, of B, and of C are among them as the numbers 442301036382, 23692678875 and 96161841776. These are there the numbers demanded.

End of the Calculus of Chances by means of Arithmetic.

Calculus of Chances by means of Algebra (pp. 46–48)

First Problem

Someone has c tickets, namely a black, b white, r red, e blue, d green, etc. One wagers to draw from them all at once at random p tickets of which n white, m black, q red, o blue, s green, etc. What chance has one to win the game?

The formula is

$$\frac{\overbrace{a.\overline{a-1}.\overline{a-2}}^{m \text{ terms}}}{1.2.3}\text{etc.} \times \frac{\overbrace{b.\overline{b-1}.\overline{b-2}}^{n \text{ terms}}}{1.2.3}\text{etc.} \times \frac{\overbrace{r.\overline{r-1}.\overline{r-2}}^{q \text{ terms}}}{1.2.3}\text{etc.}\times$$

$$\frac{\overbrace{e.\overline{e-1}.\overline{e-2}}^{o \text{ terms}}}{1.2.3}\text{etc.} \times \frac{\overbrace{d.\overline{d-1}.\overline{d-2}}^{s \text{ terms}}}{1.2.3}\text{etc.}$$

the whole divided by

$$\frac{c.\overline{c-1}.\overline{c-2}}{1.2.3}\text{etc.}$$

The total number of terms of this denominator must be equal to $m+n+q+o+s+$etc., that is to p.

First Example

One wagers to draw 4 tickets or 4 cards from a deck of 40 pieces, of which 10 of each color or of each kind, in such manner that one obtains a piece of each color or of each kind. This is the third of the last 5 Problems of Mr. C. Huygens and a particular case of the Problem considered. Indeed, one has here $a+b+r+e$, that is $c = 40$; a, b, r and e are each equal to 10, m, n, q and o to 1, p to 4. One has therefore

$$\frac{10.10.10.10}{\frac{40.39.38.37}{1.2.3.4}} \text{ or } \frac{1000}{9139},$$

so that one is able to wager 1000 against 8139.

Fourth Example

But if one had only some black tickets and some white tickets, and if $a+b$ or $c = 12$, $a = 8$, $b = 4$, $n = 3$ and $m = 4$, this would be the fourth of the last 5 Problems of

MR. C. HUYGENS. One has then:

$$\frac{\cancel{8}.\cancel{7}.\cancel{6}.5.\frac{\cancel{4}.\cancel{3}.\cancel{2}}{1.\cancel{2}.\cancel{3}}.\cancel{5}.\cancel{6}.7}{\cancel{1}\cancel{2}.11.\cancel{1}\cancel{0}.9.\cancel{8}.\cancel{7}.\cancel{6}} \text{ or } \frac{35}{99}.$$

One is able therefore to set 35 against 64.

Third Problem (pp. 58–62)

An undetermined number of players play with some dice. A will win the stake if he casts a determined number of points. B will win the stake if he casts another determined number of points; likewise for C, D, etc. A will begin by casting i times, B k times, C l times, D m times, E n times, F o times, etc. Next A will cast r times, B s times, C t times, D v times, E w times, F x times, etc., and one will continue in the same manner until one of the players will have won. What will be the chance of each of them?

We suppose that A must cast 12 points with 2 ordinary dice. Then, if he consents to not cast one time, one must give to him $\frac{1}{36}$ of the stake. There will remain then $\frac{35}{36}$. Put this number $= a$. If B must cast 11 points, one must offer to him $\frac{1}{18}$ of that which remains of the stake if he consents to permit passing his turn. There will remain then $\frac{17}{18}$. Put this number $= b$, and likewise $c =$ the corresponding number for C, d and e the corresponding numbers for D and E respectively.

We will resolve here a problem of 3 players; by aid of our solution where one will see easily how one is able to take a greater number of players to infinity. We have seen in the preceding problem that the i coups that A is able to make at the beginning have for him the value $1 - a^i$. For the other players, if A renounces play and if one pays to him the indemnity to which he has right, there remains a^i. After that B will make k coups. If the total stake is 1, his part according to the preceding problem is $1 - b^k$. Since there remained a^i, one will find therefore that the value of the coups that B is able to make at the beginning is $a^i - a^i b^k$. Subtract this number from a^i, the stake which remains; there will remain therefore, after one will have paid to B the indemnity to which he has right, $a^i b^k$. For l coups C must receive $1 - c^l$, when the stake is 1; but since it is $a^i b^k$, one finds that the coups of C have a value $a^i b^k - a^i b^k c^l$. The rest of the stake is then $a^i b^k c^l$. The r coups that A must make next, have the value $a^i b^k c^l(1 - a^r)$ according to that which precedes. If we subtract this from $a^i b^k c^l$, there remains $a^i b^k c^l a^r$. After that B casts s times. His part, according to that which precedes, is therefore [$a^i b^k c^l(a^r - a^r b^s)$. If we subtract this from $a^i b^k c^l a^r$, there remains $a^i b^k c^l a^r b^s$. The part of C, who casts t times, is therefore] $a^i b^k c^l(a^r b^s - a^r b^s c^t)$.[1] And as the parts which return to A, B, and C respectively for the coups that they must make each time, must have among them to infinity the same ratios as when the concern is with the first r coups of A, with the first s coups of B and of the first t coups of C, it follows that it suffices to divide $a^i b^k c^l$ in ratio of the following numbers.

A	$a^i b^k c^l \times (1 - a^r)$,
B	$a^i b^k c^l \times (a^r - a^r b^s)$,
C	$a^i b^k c^l \times (a^r b^s - a^r b^s c^t)$.
Sum	$a^i b^k c^l \times (1 - a^r b^s c^t)$.

The parts of the players will be therefore

[1]A gap in the text has been filled.

$$\begin{cases} \text{A} & \frac{a^i b^k c^l \times (1-a^r)}{1-a^r b^s c^t}; \\ \text{B} & \frac{a^i b^k c^l \times (a^r - a^r b^s)}{1-a^r b^s c^t}; \\ \text{C} & \frac{a^i b^k c^l \times (a^r b^s - a^r b^s c^t)}{1-a^r b^s c^t}. \end{cases}$$

And by adding to these numbers the part which returns to each of them because of the previous coups (the stake being 1), one finds for the total part of each player

$$\begin{cases} \text{A} & 1 - a^i + \frac{a^i b^k c^l \times (1-a^r)}{1-a^r b^s c^t}; \\ \text{B} & a^i - a^i b^k + \frac{a^i b^k c^l \times (a^r - a^r b^s)}{1-a^r b^s c^t}; \\ \text{C} & a^i b^k - a^i b^k c^l + \frac{a^i b^k c^l \times (a^r b^s - a^r b^s c^t)}{1-a^r b^s c^t}. \end{cases}$$

But in order to find the part which remains to each, when the number of players is any, it is necessary to note that the chance of each player is composed of an entire part and a fraction and that the entire part comprehends two terms, of which the first is positive and the second negative. The second term must contain a number of factors equal to the number of order of the player of whom one wishes to calculate the chance. The exponents must be the number of coups that the players make at the beginning. If one divides this term by the number which gives the remainder corresponding to the aforenamed player, this remainder being raised to a power equal to the number of coups that this player makes at the beginning, one has the first term. The factors themselves are the remainders a, b, c, d, e, etc. In order to find now the fraction, set in the numerator all the remainders a, b, c, d, e, etc. The exponents are the numbers i, k, l, m, n, etc. of the coups that the players are able to make at the beginning. Add as multiplier the entire part; only the exponents, instead of being equal to the numbers of the coups that the players are able to make at the beginning, must be equal to those of the regular coups, that is to r, s, t, etc. Set then in the denominator unity and subtract the first factors of the numerator from it, but by changing the exponents into r, s, t, etc. the numbers of the regular coups. You will have thus the part demanded.

If there are 6 players, we find immediately by this rule for the part of the fifth player E

$$a^i b^k c^l d^m - a^i b^k c^l d^m e^n + \frac{a^i b^k c^l d^m e^n f^o \times \overline{a^r b^s c^t d^v - a^r b^s c^t d^v e^w}}{1 - a^r b^s c^t d^v e^w f^x}$$

If one supposes that the players do not make the preliminary coups of which we have spoken, the entire part of the formula disappears; one has then $i = k = l$, etc. $= 0$. One has therefore then

$$\begin{array}{ll} \text{A} & 1-a^r \\ \text{B} & \quad +a^r - a^r b^s \\ \text{C} & \qquad +a^r b^s - a^r b^s c^t \\ \text{D} & \qquad\quad +a^r b^s c^t - a^r b^s c^t d^v \\ \text{E} & \qquad\qquad +a^r b^s c^t d^v - a^r b^s c^t d^v e^w \\ \text{F} & \qquad\qquad\qquad +a^r b^s c^t d^v e^w \\ & \qquad\qquad\qquad -a^r b^s c^t d^v e^w f^x. \end{array}$$

Here are the chances that they have with respect to one another. But if one wishes to know here, as in the formula which corresponds to the previous case, the part of the stake which returns to each of them, one is able to consider the numbers that we just wrote as the numerators of a fraction, of which the denominator is unity diminished by $a^r b^s c^t$, etc. The number of terms must be equal to the one of the players.

Third Example.

Suppose that A and B play as above with 2 ordinary dice with the condition that A will win if he casts 7 and B if he casts 6 points, that they make alternately 2 coups one after the other, A first, B next and that B will be able to begin by making a coup. This is here the first of the last 5 problems proposed by MR. C. HUYGENS, of which the solution is a particular solution comprehended in the preceding formula. Indeed, the entire part is null and $i=0$, the formula is therefore in the case considered $\frac{b^k\times\overline{1-a^r}}{1-a^rb^s}$, where $a=\frac{5}{6}$, $b=\frac{31}{36}$, $k=i$, $r=2$, and $s=2$. One finds therefore for the part of A $\frac{12276}{22631}$ and for that of B $\frac{10355}{22631}$.

Ninth Problem. (pp. 90–93)

An undetermined number of players, A, B, C, D etc. take p tickets, namely a black tickets and b white tickets and play with these tickets with this condition that the one will win the game who will have drawn first blindly a white ticket from them. A draws first, next B, next C and consecutively by turn of role. One demands what is the chance of each player.

One would be able to understand the proposed question in two ways. In first place one is able to be figured as if A, B, C etc. come to draw each a black ticket, these tickets will not be restored (that which is to the advantage of the following players), but one is able equally to interpret the question in such manner that the one who draws a bad ticket restores it immediately. We will speak first of the first case, and next of the second.

If one supposes that there are s players in all, and that the rank of the player of which one desires to know the part is r, if one puts besides $p-r=q$, $q-s=h$, $q-2s=i$, $q-3s=k$, etc., one is able to find by the following formula the ratios of the chances of the different players. One is able to take for r successively the values $1, 2, 3$ etc. to s.

$$\frac{q\times\overline{q-1}\times\overline{q-2}\times\overline{q-3}\times\overline{q-4}}{1\times2\times3\times4\times5}\text{ etc. }+\frac{h\times\overline{h-1}\times\overline{h-2}\times\overline{h-3}\times\overline{h-4}}{1\times2\times3\times4\times5}$$
$$\text{etc. }+\frac{i\times\overline{i-1}\times\overline{i-2}\times\overline{i-3}\times\overline{i-4}}{1\times2\times3\times4\times5}\text{ etc. }+$$
$$+\frac{k\times\overline{k-1}\times\overline{k-2}\times\overline{k-3}\times\overline{k-4}}{1\times2\times3\times4\times5}\text{ etc. }+\text{ etc.}$$

Each product must have $(b-1)$ factors.

If one wishes to know the part of the stake which returns to each player, one is able to take the preceding expression for numerator. The denominator must be then the following fraction, of which the numerator and the denominator are the products formed by a number of terms such that the last term is equal to $(b+1)$ or to $(a+1)$:

$$\frac{p\times\overline{p-1}\times\overline{p-2}\times\overline{p-3}\times\overline{p-4}\times\overline{p-5}}{1\times2\times3\times4\times5\times6}\text{ etc.}$$

First Example.

We suppose that there are three players A, B and C, having 12 tickets, namely 4 white tickets and 8 black tickets. The problem is then identical to the second of the last 5 Problems that Mister C. HUYGENS has proposed to the amateurs. One has in this case $p=12$, $a=8$, $b=4$, and putting $r=1,2,3$, one finds that which follows

$$\begin{array}{lll} \text{A} & \frac{11\times10\times9}{1\times2\times3}+\frac{8\times7\times6}{1\times2\times3}+\frac{5\times4\times3}{1\times2\times3} & \text{or } 231, \\ \text{B} & \frac{10\times9\times8}{1\times2\times3}+\frac{7\times6\times5}{1\times2\times3}+\frac{4\times3\times2}{1\times2\times3} & \text{or } 159, \\ \text{C} & \frac{9\times8\times7}{1\times2\times3}+\frac{6\times5\times4}{1\times2\times3}+\frac{3\times2\times1}{1\times2\times3} & \text{or } 105. \end{array}$$

By dividing the whole by 3, one finds that the chances of A, of B and of C are among them as the numbers 77, 53 and 35.

Sixth Example.

If A, in the Problem of MR. HUYGENS, wishes to wager that a person will have won the game yet, when one will have already chosen 3 tickets, one will find that B must set $2\frac{13}{14}$ florins against the florin of A.

Fourteenth Problem (pp. 108–110)

A and B have each a certain number of pieces of money. One will cast some dice of which some faces are in favor of A: when these present themselves, B must give a piece of money to A. Some other faces are in favor of B: when these present themselves, B receives from A a piece of money. The one who will be the first in possession of all the pieces of money, will win the game. One demands what is the part of each player.

Suppose that A and B have respectively r and s pieces of money. Put $r+s=d$ and suppose besides that A has b chances to win a piece of money, and B c chances.

Put $x =$ the part of A when he has 1 piece of money, $z =$ his part when he has 2 of them, and $y =$ his part when he has 3 of them, the number of pieces that B possesses being any.

Chances

$$\begin{array}{lcc} & b-z & \left\{\begin{array}{c} bz \\ 0 \end{array}\right. \\ A\ a & c-0 & \\ \hline & b+z & bz \end{array}$$

$$b+z\text{—}$$
$$\frac{bz}{b+z} = x$$
$$\text{—}b+z$$
$$b\frac{bz=bx+cx}{x=x+\frac{c}{b}x}$$

Chances

$$\begin{array}{cc} b-y & \left\{\begin{array}{c} by \\ cx \end{array}\right. \\ c-x & \\ \hline b+c & by+cx \end{array}$$

$$b+c\text{—}$$
$$\frac{by+cx}{+c} = z = x + \frac{c}{b}x$$
$$\text{——}b+c$$
$$by+cx = bx+cx+\frac{bc+cc}{b}x$$

or

$$y = x + \frac{c}{b}x + \frac{cc}{bb}x.$$

And by continuing thus one will find:

$$\text{If A has}\left\{\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\right\}\begin{array}{c} \text{pieces of} \\ \text{money, his} \\ \text{part is} \end{array}\left\{\begin{array}{l} x \\ x+\frac{c}{b}x \\ x+\frac{c}{b}x+\frac{cc}{bb}x \\ x+\frac{c}{b}x+\frac{cc}{bb}x+\frac{c^3}{b^3}x \\ x+\frac{c}{b}x+\frac{cc}{bb}x+\frac{c^3}{b^3}x+\frac{c^4}{b^4}x. \end{array}\right.$$

When A has all the pieces of money, he wins the total stake. The total stake is therefore equal to a sum of terms of which the number is equal to the one of the pieces of money, so that his part is expressed by the following fraction

$$\frac{x + \frac{1}{q}x + \frac{cc}{bb}x + \frac{c^3}{b^3}x + \frac{c^4}{b^4}x + \text{etc. } (r \text{ terms})}{x + \frac{c}{b}x + \frac{cc}{bb}x + \frac{c^3}{b^3}x + \frac{c^4}{b^4}x + \text{etc. } (d \text{ terms})}$$

By making the fraction vanish and dividing the whole by x, one obtains

$$\frac{b^{d-1} + b^{d-2}c + b^{d-3}cc + b^{d-4}c^3 + b^{d-5}c^4 + \text{etc. } (r \text{ terms})}{b^{d-1} + b^{d-2}c + b^{d-3}cc + b^{d-4}c^3 + b^{d-5}c^4 + \text{etc. } (d \text{ terms})}$$

The numerator and the denominator both form a geometric progression. The sum of the first is $\frac{b^d - c^r b^s}{b-c}$, and that of the second $\frac{b^d - c^d}{b-c}$. The part of A is therefore $\frac{b^d - c^r b^s}{b^d - c^d}$, and that of B $\frac{c^r b^s - c^d}{b^d - c^d}$.

First Example.

Two persons A and B play with 2 dice. A has 19 pieces of money, B has 5 of them. A must give a piece of money to B when one casts 10 points, and B a piece of money to A when one casts 11. What is the chance of each of them?

One will find that A has 245237169537 chances to lose and 37175549728 chances to win, so that in order to play against B with neither advantage nor disadvantage he should set 1 into the game and B a little less than 7.

When both have the same number of pieces of money, that is when $r = s$ and $d = 2s$, the part of A is $\frac{b^{2s} - c^s b^s}{b^{2s} - c^{2s}}$. If one divides the whole by $b^s - c^s$, there comes $\frac{b^s}{b^s - c^s}$. And for the part of B one finds next $\frac{c^s}{b^s + c^s}$.

Second Example.

Having taken each 12 pieces of money, A and B play with 3 dice with the condition that when one casts 11 points, A must give a piece of money to B, but when one casts 14 of them, B must give to one to A and that the one will win the game who first will be in possession of all the pieces of money. What chance to win will they both have? This is the last of the Problems posed by Mr. C. HUYGENS.

With 3 dice one is able to make 27 coups of 11 points and 15 coups of 14 points. The chances that they have to win a piece of money are therefore between them as 5 is to 9. The part of A is therefore 5^{12}, that of B 9^{12}, in other terms the chance of A is to that of B as 244140625 is to 282429536481.