DISQUISITIO ELEMENTARIS CIRCA CALCULUM PROBABILIUM*

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Many scattered analytic meditations exist here and there concerning the calculation of Probabilities, the resolve to examine which things is not here. While generally some consider particular questions, the greatest Geometers Laplace and Lagrange have attempted to treat this theory more generally, deriving assistance out of the most intimate inmost parts of the calculation of integers, and indeed they have secured extraordinary fruits thereupon. But while the whole theory of Probabilities may be supported by simple and obvious principles, which require nothing in general other than the theory of combinations, and most difficulties may be meditated upon with regard to enumerating and distinguishing cases, it had seemed because of the fact, to treat the same more general questions by an elementary method, without any foreign assistance. Of which effort these pages embrace the first specimen, they contain of course elementary solutions of the more general Problems which the most illustrious man Lagrange gave solutions in *Nouveaux Mémoires de l'Académie royale des Sciences et Belles-Lettres de Berlin*[1] for the year 1775. If these will not have displeased the Geometers, elsewhere then, I will put forward clarifications of the same kind by themselves with God helping.

Problem 1.[2]

1. A gamester wagers to bring forth all an event b times neither more nor less, but exactly a casts are permitted, moreover the Probability of each and every cast is p, the lot of the gamester is sought.

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[1] *Translator's note*. The full title of this paper is "Recherches sur les suites récurrentes dont les termes varient de plusiers manieres différentes, ou sur l'integration des équations linéares aux différences finies et partielles; et sur l'usage de ces équations dans la théorie des hasards." pp. 183-272.

[2] *Translator's note*. This is a Corollary to Problem I of Lagrange. See Sections 49 & 50 of the paper cited.

Solution.

The Probability of bringing forth the event b times following each other is p^b. The Probability of bringing forth the other event $a-b$ times following each other is $= (1-p)^{a-b}$. The Probability of bringing forth the event b times precisely when there are a casts permitted is $p^b(1-p)^{a-b}$. This Probability must be multiplied by the number of combinations a of quantities according to exponent b, it is by

$$\frac{a(a-1)\dots(a-b+1)}{1.2.3\dots b} = \frac{1.2\dots a}{1.2\dots b.1.2\dots(a-b)} = \frac{(b+1)(b+2)\dots a}{1.2.3\dots a-b}.$$

The sought Probability will be therefore

$$\frac{(b+1)(b+2)\dots a}{1.2.3\dots a-b}p^b(1-p)^{a-b}.$$

The Learned de la Grange finds the same, if in his formula you correct the typographical error, namely it contains in the denominator the factor $a-b+1$ which should be erased.

Problem 2.[3]

2. A gamester wagers to bring forth some event b times, or more, but exactly a casts are permitted, moreover the Probability of each and every cast is p, the lot of the gamester is sought.

Solution.

The sought Probability is the sum of the following Probabilities: the Probabilities of bringing forth the event precisely b times; the Probabilities of bringing forth the event precisely $b+1$ times; the Probabilities of bringing forth the event precisely $b+2$ times... the Probabilities of bringing forth the event precisely a times. Moreover it follows from the preceding Problem, when a casts are permitted; the Probability of bringing forth the event 1 time is precisely

$$= ap(1-p)^{a-1}.$$

The Probability of bringing forth the event 2 times is precisely

$$= \frac{a(a-1)}{1.2}p^2(1-p)^{a-2};$$

the Probability of bringing forth the event precisely 3 times is

$$= \frac{a(a-1)(a-2)}{1.2.3}p^3(1-p)^{a-3}$$

[3] *Translator's note*. The statement of this problem is the same as that of Problem I in the paper of Lagrange. See his section 49.

and thus successively. Therefore the Probability of bringing forth the event precisely b times is

$$= \frac{(b+1)(b+2)\dots a}{1.2.3\dots a-b} p^b (1-b)^{a-b}.$$

The Probability of bringing forth the event precisely $b+1$ times is

$$= \frac{(b+2)(b+3)\dots a}{1.2.3\dots a-b+1} p^{b+1} (1-b)^{a-b-1};$$

the Probability of bringing forth the event precisely $b+2$ times is

$$= \frac{(b+3)(b+4)\dots a}{1.2.3\dots a-b+2} p^{b+2} (1-b)^{a-b-2}$$

and thus successively. Next the Probability of bringing forth the event precisely a times is $= p^a$. The sought Probability will be therefore $=$

$$\begin{aligned}
&p^a + ap^{a-1}(1-p) + \frac{a(a-1)}{2} p^{a-2}(1-p)^2 \dots \\
&\quad + \frac{a(a-1)\dots(b+1)}{1.2\dots(a-b)} p^b (1-p)^{a-b} = \\
&\quad p^b \left(p^{a-b} + ap^{a-b-1}(1-p) + \frac{a(a-1)}{2} p^{a-b-2}(1-p)^2 \dots \right. \\
&\quad \left. + \frac{a(a-1)\dots(b+1)}{1.2\dots(a-b)} p^b (1-p)^{a-b} \right).
\end{aligned}$$

The form of this formula was different from it, which the Celebrated Lagrange gives, yet returns to the same, as is easily proved *a posteriori*, it was long and wearisome to compare these formulas to one another *a priori*. It will be short and simple to deduce the formula of the Learned Lagrange out of the following reasoning, because chiefly it will be useful to us in the future. The sought Probability is able to be considered as the sum of the following Probabilities; of bringing forth the event b times following themselves, with $b+1$ casts permitted, of bringing forth the event b times, with $b+2$ casts permitted, of bringing forth the event b times, and thus successively, finally with a casts permitted, of bringing forth the event b times, yet by subtracting from the cases of each new Probability the case of the preceding Probabilities lest the same cases are selected several times. Moreover the Probability of bringing forth the event b times following themselves is p^b. The Probability of bringing forth the event b times precisely, with $b+1$ casts permitted is $(b+1)p^b(1-p)$. From these cases should be subtracted the case in which the event is brought forth with the first b casts, which case now was introduced into the calculation, therefore there remains $bp^b(1-p)$. The probability of bringing forth the event b times, with $b+2$ casts permitted is $\frac{(b+2)(b+1)}{1.2} p^b (1-b)^2$, the preceding cases should be subtracted. Therefore there remains

$$\left(\frac{(b+2)(b+1)}{1.2} - (b+1) \right) p^b (1-p)^2 = \frac{b(b+1)}{1.2} p^b (1-p)^2.$$

In like manner the Probability will be obtained of bringing forth the event b times, with $b+3$ casts permitted =

$$\left(\frac{(b+3)(b+2)(b+1)}{1.2.3}-\frac{b(b+1)}{1.2}-(b+1)\right)p^b(1-p)^3=\frac{b(b+1)(b+2)}{1.2.3}p^b(1-p)^3$$

and thus successively up to

$$\frac{b(b+1)\dots(a-1)}{1.2.3\dots(a-b)}p^b(1-p)^{a-b}.$$

Therefore the sought Probability will be

$$p^b\left(1+b(1-p)+\frac{b(b+1)}{2}(1-p)^2\dots+\frac{b(b+1)\dots(a-1)}{1.2.3\dots(a-b)}(1-p)^{a-b}\right)$$

This is the formula which the Learned Lagrange deduced from his calculations.

Scholium.

3. Problems 3, 4 and 5 of the extraordinary work of the Celebrated Moivre to which the title is recorded as *Doctrine of Chances* are able to be solved with help of this Problem. In Problem 3 the Celebrated Man asks with how many casts it is necessary, in order that he is able to predict some event to arrive or not with equal lot, let p be the Probability of the sought event, q the Probability of the contrary event, x the sought number of casts, if there is in our formula $a=x$, $b=x$, $p=q$, the sought Probability is q^x and by the conditions of the Problem we have $q^x=\frac{1}{2}$, therefore $x=-\frac{\ln 2}{\ln q}$. Let be made with Moivre $q=\frac{a}{a+b}$, we have $x=\frac{\ln 2}{(a+b)-\ln b}$ which is the formula of Moivre.[4] This solution in which the Probability of the event is sought the other not going to arrive is briefer in which it must happen $a=x$, $b=1$, $p=p$. We have in this case

$$p(1+p+q^2+q^3\dots+q^{x-1})=\frac{1}{2}$$

(here there is $q=1-p$) or by making the sum of the geometric progression,

$$p\left(\frac{1-q^x}{1-q}\right)=\frac{1}{2},$$

or $1-q^x=\frac{1}{2}$ or $q^x=\frac{1}{2}$ as above. The comparison of these solutions gives the sum of the geometric progression. For since there is

$$(1-q)(1+q+q^2+q^3\dots+q^{x-1})=1-q^x$$

there will be made

$$1+q+q^2+q^3\dots+q^{x-1}=\frac{1-q^x}{1-q}.$$

[4] *Translator's note*. The formula is in error. It should read $x=\frac{\ln 2}{\ln(a+b)-\ln b}$.

In the 4th Problem Moivre seeks, in how many casts it is necessary, in order that he is able to predict with equal lots, some event to arrive two times or not. Let p be the Probability of the sought event, q the Probability of the contrary event. Thus this problem is able to be proposed: The Probability is sought of some event of which the Probability is q, with x casts permitted going to arrive at least $x-1$ times. Let be made in our formula $a = x$, $b = x - 1$, $p = q$, we will have

$$q^{x-1}[1 + (x-1)(1-q)] = \frac{1}{2}.$$

Let with Moivre be made $p = \frac{a}{a+b}$, $q = \frac{b}{a+b}$, it makes

$$\frac{b^x}{(a+b)^{x-1}}\left(1 + \frac{(x-1)a}{a+b}\right) = \frac{b^{x-1}}{(a+b)^{x-1}}\left(\frac{b+ax}{a+b}\right) = \frac{b^x + axb^{x-1}}{(a+b)^x} = \frac{1}{2}$$

which is the Moivrean equality. With regard to direct solution the Probability is sought of some event, with x casts permitted, to arrive at least 2 times. Let there be in our formula $a = x$, $b = 2$, the equality will be had

$$p^2[1 + 2q + 3q^2 + 4q^3 \ldots + (x-1)q^{x-2}] = \frac{1}{2}.$$

But it is agreed by known methods to be

$$1 + 2q + 3q^2 \ldots + (x-1)q^{x-2} = \frac{1 - q^{x-1} - (x-1)q^{x-1}(1-q)}{(1-q)^2}.$$

Therefore the equality will be made $q^x[1 + (x-1)(1-q)] = \frac{1}{2}$ as above. The comparison of the two solutions gives the sum of the series, of which the terms follow a geometric progression but the coefficients of the terms follow an arithmetic progression. For while there is

$$(1-q)^2[1 + 2q + 3q^2 + 4q^3 \ldots + (x-1)q^{x-2}] = 1 - q^{x-1} - (x-1)q^{x-1}(1-q),$$

therefore

$$1 + 2q + 3q^2 \ldots + (x-1)q^{x-2} = \frac{1 - q^{x-1}}{(1-q)^2} - \frac{(x-1)q^{x-1}}{1-q}$$

as it is agreed.

In the 5th Problem it is asked by Moivre, with how many casts it is necessary, in order that he is able to predict with equal lots, any event to be brought forth three times or not. Let p be the Probability of the sought event, q the Probability of the contrary event. Thus the Problem is able to be proposed. The Probability is sought of any event of which the Probability is q, x casts permitted to arrive at least $x - 2$ times. Let be made in our formula $a = x$, $b = x - 2$, $p = q$, the equality will be,

$$q^{x-2}[1 + (x-2)(1-q) + \frac{(x-2)(x-1)}{2}(1-q)^2] = \frac{1}{2}.$$

Let there be with Moivre $p = \frac{a}{a+b}$, $q = \frac{b}{a+b}$, there will be

$$\frac{b^{x-2}}{(a+b)^x}[bb + xab + \frac{x(x-1)}{2}aa] = \frac{1}{2};$$

in direct solution of the Problem the equality arrives,

$$p^3[1 + 3q + 6q^2 + 10q^3 \ldots + \frac{(x-1)(x-2)}{1.2}q^{x-3}] = \frac{1}{2}.$$

The comparison of the two solutions gives the sum of the series, of which the terms follow a geometric progression, but the coefficients follow a progression of triangular numbers. And indeed by comparing the two formulas it produces,

$$1 + 3q + 6q^2 + 10q^3 \ldots + \frac{(x-1)(x-2)}{1.2}q^{x-3}$$
$$= \frac{1-q^{x-2}}{(1-q)^3} - \frac{(x-2)q^{x-2}}{(1-q)^2} - \frac{(x-2)(x-1)}{1.2.(1-q)}q^{x-2}$$

as it is agreed, the law of progression is evident. It is apparent in the case of 4 events to produce the equation

$$q^{x-3}[1+(x-3)(1-q)+\frac{(x-3)(x-2)}{1.2}(1-q)^2+\frac{(x-3(x-2)(x-1)}{1.2.3}(1-q)^3] = \frac{1}{2}$$

the law of progression is evident.

Problem 3.

4. Two events are able to arrive in any one cast, of which the Probabilities are p and q, the lot of the gamester is sought, who wagers to bring forth the other event b times, before the first will appear a times, with an indeterminate number of casts permitted.

Solution.

The solution of this Problem coincides with the second Problem preceding. For the Probability of the gamester is the sum of the following Probabilities: of the Probability the other event to arrive b times first, which is $= q^b$; of the Probability the same event to arrive b times, with $b + 1$ casts permitted, by subtracting the preceding cases, which is bq^pp; of the Probability the same event to arrive b times, with $b + 2$ casts permitted, by subtracting the preceding cases, which is $\frac{b(b+1)}{2}q^bp^2$; of the Probability the same event to arrive b times, with $b + 3$ casts permitted, by subtracting the preceding cases, which is $\frac{b(b+1)(b+2)}{2.3}q^bp^3$ and thus successively; and next the Probability the same event to arrive b times, with $b+a-1$ casts permitted, by subtracting the preceding cases, which Probability is obvious from the law of progression $\frac{b(b+1)(b+2)\ldots(b+a-2)}{1.2.3\ldots(a-1)}q^bp^{a-1}$. By making the sum of these Probabilities, the sought Probability appears

$$= q^b\left(1 + bp + \frac{b(b+1)}{2}p^2 + \frac{b(b+1)(b+2)}{2.3}p^3 \ldots + \frac{b(b+1)(b+2)\ldots(b+a-2)}{1.2.3\ldots a-1}p^{a-1}\right).$$

This same the Celebrated Lagrange finds in his first solution of this Problem.

Scholium.

5. The first two Problems of Moivre are able to be solved with the aid of this Problem and indeed in a simpler manner. Let the two gamblers be A and B, but it is agreed by trial the gamester A to be of such skill, that of three games he is able to concede two to B, the lot of gambler A is sought for any one game, or the ratio of skill. In this Problem the unknown quantities are p and q. Let there be $q = \frac{1}{x}$, $p = \frac{x-1}{x}$, by the condition of the Problem this is, in order that A is able to promise with equal right, himself is going to be the victor in three games sooner than B is the victor in one. Let there be $a = 1$, $b = 3$. The sought Probability is $= \frac{1}{x^3}$, therefore the equality produces $\frac{1}{x^3} = \frac{1}{2}$ or $x = \sqrt[3]{2}$. Hence the proportion appears $q : p = 1 : x - 1 : 1 : \sqrt[3]{2} - 1$, as Moivre reports.

If out of the three games A is able to concede one game to gambler B, the lot of the gambler A is sought for any one game. By the condition of the Problem this is, in order that A is able to promise with equal right, himself going to be the victor in three games sooner than B will be victor in two. By the same values having been used for q and p, and by supposing $a = 2$, $b = 3$, the sought Probability is

$$\frac{1}{x^3}\left(1 + \frac{3(x-1)}{x}\right) = \frac{4x-3}{x^4} = \frac{1}{2}.$$

Therefore the equality appears $x^4 = 8x + 6$; $x = 1.625$ as nearly. Therefore $q : p = 1 : 0.625 = 8 : 5$, as Moivre reports.

Problem 4.[5]

6. The three events P, Q, R are able to arrive in any one cast, of which the Probabilities are respectively p, q, r; the lot of the gambler is sought, who wagers to bring forth the event R, c times, before the event Q will arrive b times, but the event P a times.

Solution.

Of this problem the type is the same as the preceding; the Probability is formed in precisely the same manner, in the same manner the coefficients of each and every term appear, indeed on account of the total of three events it is necessary for some further calculations. But first, the gambler wins, if he will bring forth event R, c times first, of which case the Probability is r^c. Second, the same gambler wins, if he will bring forth the event c times, $c+1$ casts permitted, of which case the Probability for event P is by the preceding Problem cr^cp, for the event Q, by the same Problem cr^cq, therefore it produces the Probability $cr^c(p+q)$. Third, the gamester wins if he will bring forth the event R, c times, $c+2$ casts permitted, of which case the Probability by the preceding Problem is $\frac{c(c+1)}{1.2}p^2$ for the event P, and $\frac{c(c+1)}{1.2}q^2$ for the event Q. In addition they should consider here jointly the events P and Q, for when $c+2$ casts are given, the event R is able to arrive c times, the event P, 1 time, the event Q, 1 time, of which case

[5] *Translator's note*. This is the Problem of Points for 3 players. See Problem VI of Moivre, 3rd edition.

the Probability is $2pq$: therefore it will produce $\frac{c(c+1)}{1.2}2pq$. Therefore the Probability of the third case will be

$$= \frac{c(c+1)}{1.2}(p^2 + 2pq + q^2) = \frac{c(c+1)}{1.2}(p+q)^2.$$

Fourth, the gambler wins, if he will bring forth the event R, c times, $c+3$ casts permitted, this event P is able to arrive 3 times, of which case the Probability is by the preceding Problem $\frac{c(c+1)(c+2)}{2.3}r^c p^3$; the event P is able to arrive 2 times, also the event Q one time, of which case the Probability is $= \frac{c(c+1)(c+2)}{2.3}3r^c p^2 q$; the event P is able to arrive 1 time, also the event Q, 2 times, of which case the Probability is $= \frac{c(c+1)(c+2)}{2.3}3r^c pq^2$, finally the event Q is able to arrive 3 times, of which case the Probability is $= \frac{c(c+1)(c+2)}{2.3}r^c q^3$. Therefore the Probability of the fourth case is

$$= \frac{c(c+1)(c+2)}{2.3}r^c(p^3 + 3p^2q + 3pq^2 + q^3) = \frac{c(c+1)(c+2)}{2.3}r^c(p+q)^3.$$

By the evident law of progression it is able to be concluded, the Probability of the case in which the gambler wagers to bring forth event R, c times with $c+n$ casts permitted, to be

$$= \frac{c(c+1)(c+2)\dots(c+n-1)}{1.2.3\dots n}r^c(p+q)^n.$$

But because generally it is able to be demonstrated with the following reckoning. With $c+n$ casts permitted the event P is able to arrive n times, of which case the Probability is

$$\frac{c\dots(c+n-1)}{1\dots n}p^n,$$

the event P is able to arrive $(n-1)$ times, also the event Q, 1 time, of which case the Probability is

$$= \frac{c\dots(c+n-1)}{1\dots n}np^{n-1}q;$$

the event P is able to arrive $n-2$ times, but the event Q, 2 times, of which case the Probability is

$$= \left(\frac{c\dots(c+n-1)}{1\dots n}\right)\frac{n(n-1)}{1.2}p^{n-2}q^2;$$

the event P is able to arrive $n-3$ times, but the event Q, 3 times, of which case the Probability is

$$= \frac{c\dots(c+n-1)}{1\dots n}\frac{n(n-1)(n-2)}{1.2.3}p^{n-3}q^3,$$

generally the event P is able to arrive $n-d$ times, but the event Q, d times, of which case the Probability is

$$= \frac{c\dots(c+n-1)}{1\dots n}\frac{n(n-1)(n-2)\dots(n-d+1)}{1.2\dots d}p^{n-d}q^d.$$

Therefore the Probability of the $(n+1)^{\text{st}}$ case will be

$$\frac{c(c+1)\dots(c+n-1)}{1\dots n}r^c(p+q)^n.$$

By making the sum of all such Probabilities the sought Probability is obtained

$$= r^c \left(1 + c(p+q) + \frac{c(c+1)}{1.2}(p+q)^2 + \frac{c(c+1)(c+2)}{1.2.3}(p+q)^3 \dots \frac{c\dots(c+n-1)}{1\dots n}(p+q)^n\right)$$

which formula the Celebrated Lagrange obtains. It is clear from the condition of the Problem, this series to be evolving, until when the exponent of the quantity p becomes $= a$, also the exponent of the quantity q becomes $= b$, such cases to be precluded. Let there be, for the sake of brevity, $c = u'$, $\frac{c(c+1)}{1.2} = u''$ etc. there will be obtained the evolving series, of which r^c is a factor,

$$\begin{array}{ll}
1+u'p+u''p^2 \cdots\cdots & +u^{(x-1)}p^{x-1} \\
+u'q+2u''pq+3u'''p^2q \cdots\cdots & +xu^{(x)}p^{(x-1)}q \\
+u''q^2+3u'''pq^2+6u'''p^2q^2 & +\frac{x(x+1)}{2}u^{(x+1)}p^{x-1}q^2 \\
\dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots & \dots\ \dots\ \dots\ \dots\ \dots \\
+u^{(c-1)}q^{(t-1)}+tu^{(t)}pq^{(t-1)}+\frac{t(t+1)}{2}u^{(t+1)}p^2q^{t-1}\cdots & +\frac{t\dots(t+x-2)}{1\dots(x-1)}u^{(x+t-2)}p^{x-1}q^{t-1}.
\end{array}$$

The coefficient of the last term is able to be put into this form also $\frac{x(x+1)\dots x+t-2}{1.2\dots t-1}$ as is clear.

Scholium.

7. With this Problem the general solution is contained in the Problem, which is called commonly of the three gamblers, of which Moivre gives an example in Prob. VI. We will treat this example, in order that it is able to be seen, by which applied method of the formula it must be done. Three gamblers, A, B, C play under this condition, that he may hold the deposited sum, who first will be the victor a certain number of times, with the probability being of one and all for each and every trial respectively a, b, c. With time elapsed, this is the state of the thing: there are lacking to the gambler C 3 coups, to gambler B 2, to gambler A 1, as they had intended the deposit, the expectation of each and every one is sought. Moreover the deposited sum is $= 1$. Here there must be

$$p = \frac{a}{a+b+c},\ q = \frac{b}{a+b+c},\ r = \frac{c}{a+b+c},\ a = 1,\ b = 2,\ c = 3$$

if the expectation of gambler C is sought. In addition not unless b must be raised to the first power, but a to appear nowhere. Therefore the formula will become

$$\frac{c^3}{(a+b+c)^3}\left(1 + \frac{3b}{a+b+c}\right) = \frac{ac^3 + 4bc^3 + c^4}{(a+b+c)^4}.$$

If the expectation of the gambler B is sought, there must be

$$p = \frac{a}{a+b+c},\ q = \frac{c}{a+b+c},\ r = \frac{b}{a+b+c},\ a = 1,\ b = 3,\ c = 2.$$

In addition not unless c must be raised to the second power, but a to appear nowhere. Therefore the formula will become

$$\frac{b^2}{(a+b+c)^3}\left(1 + \frac{2c}{a+b+c} + \frac{3c^2}{(a+b+c)^2}\right)$$
$$= \frac{a^2b^2 + b^4 + 6b^2c^2 + 2ab^3 + 4abc^2 + 4bc^3}{(a+b+c)^4},$$

if the expectation of the gambler A is sought, there must be

$$p = \frac{c}{a+b+c},\ q = \frac{b}{a+b+c},\ r = \frac{a}{a+b+c},\ a = 3,\ b = 2,\ c = 1.$$

In addition not unless c must be raised to the second power, but not unless b to the first. Therefore the formula becomes

$$\frac{a}{a+b+c}\left(1 + \frac{b+c}{a+b+c} + \frac{2bc+c^2}{(a+b+c)^2} + \frac{3bc^2}{(a+b+c)^3}\right)$$
$$= \frac{a^4 + 4a^3b + 5a^2b^2 + 2ab^3 + 4a^3c + 12a^2bc + 8ab^2c + 6a^2c^2 + 3ac^3 + 12abc^2}{(a+b+c)^4},$$

Moivre obtains the same formulas. If these three expectations are added, there will be obtained,

$$\frac{a^4 + 4a^3b + 6a^2c^2 + 4ab^3 + 4a^3c + 12a^2bc + 12ab^2c + 6a^2c^2 + 12abc^2 + 4ac^3 + b^4 + 6b^2c^2 + 4b^3c + c^4}{(a+b+c)^4}$$
$$= \frac{(a+b+c)^4}{(a+b+c)^4} = 1,$$

what the nature of the thing demands.

Problem 5.[6]

8. With any one cast four events P, Q, R, S are able to arrive, of which the Probabilities are respectively $p,\ q,\ r,\ s$, the lot of the gambler, who wagers to bring forth the event P α times, before Q will arrive β times, R γ times, S δ times.

Solution.

It is apparent this problem is able to be solved in precisely the same way as the preceding, indeed so that the more general solution may proceed we seek the coefficient of the term $s^\alpha p^l q^m r^n$. It follows from the preceding Problem and the reasoning of Problem 2 the total coefficient of the term to be

$$= \frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+l+m+n-1)}{1.2\ldots l+m+n},$$

this is the coefficient of the term, by which the Probability is expressed of the event δ to arrive α times, with $\alpha + l + m + n$ trials permitted, if it is multiplied by

$$\frac{l(l+1)\ldots(l+m+n)}{1.2.3\ldots m.1.2.3\ldots n}.$$

The sought term will be therefore,

$$\frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+l+m+n-1)}{1.2\ldots(l+m+n)} \cdot \frac{(l+1)(l+2)\ldots(l+m+n)}{1.2.3\ldots m.1.2.3\ldots n} s^\alpha p^l q^m r^n$$

[6] *Translator's note*. This is the Problem of Points for 4 players.

and this solution for its own sake is extended to any number of events. If therefore in accordance with this kind the events A, B, C, D etc. are able to arrive of which the respective Probabilities are a, b, c, d etc. and the lot of the gambler is sought, who wagers to bring forth the event A α times before B will arrive β times, C γ times, D δ times etc. the term of the formula $a^\alpha b^l c^m d^n$ will be with regard to kind

$$\frac{\alpha(\alpha+1)\ldots(\alpha+l+m+n\,\text{etc.}-1)}{1.2\ldots(l+m+n\,\text{etc.})} = \frac{(l+1)(l+2)\ldots(l+m+n\,\text{etc.})}{1.2\ldots m.1.2\ldots n.1.2\,\text{etc.}} a^\alpha b^l c^m d^n \text{ etc.}$$

but all whole numbers from 0 to $\beta-1$, $\gamma-1$, $\delta-1$ respectively must be substituted successively for l, m, n etc. The Celebrated Lagrange obtains the same solution. Thus in an easy manner the general expectation of the gambler is able to be deduced. For out of the Newtonian theorem there appears

$$\begin{aligned}(b+c+d)^{l+m+n} =& b^{l+m+n} + (l+m+n)b^{l+m+n-1}(c+d)\\ &+ \frac{(l+m+n)(l+m+n-1)}{2} b^{l+m+n-2}(c+d)^2 \ldots \text{ etc.}\\ &+ \frac{(l+m+n)(l+m+n-1)\ldots(l+1)}{1.2.3\ldots(m+n)} b^l (c+d)^{m+n} \text{ etc.}\end{aligned}$$

There appears likewise

$$\begin{aligned}(c+d)^{m+n} =& c^{m+n} + (m+n)c^{m+n-1}d\\ &+ \frac{(m+n)(m+n-1)}{2} c^{m+n-2}d^2 \text{ etc.}\\ &+ \frac{(m+n)(m+n-1)\ldots(m+1)}{1.2.3\ldots n} c^m d^n.\end{aligned}$$

Therefore in the formula $(b+c+d)^{l+m+n}$ the coefficient of the term $b^l c^m d^n$ is

$$\begin{aligned}&\frac{(l+m+n)(l+m+n-1)\ldots(l+1)}{1.2.3\ldots(m+n)} \frac{(m+n)(m+n-1)\ldots(m+1)}{1.2.3\ldots n}\\ &= \frac{(l+1)(l+2)\ldots(l+m+n)}{1.2.3\ldots m.1.2.3\ldots n}.\end{aligned}$$

Therefore the Probability to bring forth the event R with $\alpha+l+m+n$ casts permitted is

$$= \frac{\alpha(\alpha+1)\ldots(\alpha+l+m+n-1)}{1.2\ldots(l+m+n)}(b+c+d)^{l+m+n},$$

and hence the expectation of the gambler in accordance with this kind is expressed through the series

$$\begin{aligned}&a^\alpha[1+\alpha(b+c+d\,\text{etc.}) + \frac{\alpha(\alpha+1)}{2}(b+c+d\,\text{etc.})^2\times\\ &+ \frac{\alpha(\alpha+1)(\alpha+2)}{2.3}(b+c+d\,\text{etc.})^3 \text{ etc.}]\end{aligned}$$

by preserving so many same terms in which the exponents of the quantity b will be less than β, the exponents of the quantity c will be less than γ, the exponents of the quantity d will be less than δ and thus successively, as the Celebrated Lagrange reports.

Scholium.

9. This Problem coincides with Problem LXIX[7] of Moivre, this contributed example is solved as a result of the fourth Problem. Let there be three gamblers A, B, C, of whom the respective Probabilities are a, b, c, to gambler A are lacking 2 coups, to the gambler B 3 coups, to the gambler C 5 coups in order that they be victors, the expectation of the gambler A is sought. In the formula of the fourth Problem there must be made $r = \frac{a}{a+b+c}$, $q = \frac{b}{a+b+c}$, $p = \frac{c}{a+b+c}$, $c = 2$, $b = 3$, $a = 5$. The exponent of the quantity b must not exceed 2, the exponent of the quantity c must not exceed 4. Therefore the formula will become,

$$\frac{a^2}{(a+b+c)^2}\left(1+\frac{2(b+c)}{a+b+c}+\frac{3(b+c)^2}{(a+b+c)^2}+\frac{4(3b^2c+3bc^2+c^3)}{(a+b+c)^3}\right.$$
$$\left.+\frac{5(6b^2c^2+4bc^3+c^4)}{(a+b+c)^4}+\frac{6(10b^2c^3+5bc^4)}{(a+b+c)^5}+\frac{7.15c^4b^2}{(a+b+c)^6}\right),$$

as Moivre reports.

Problem 6.[8]

9. Let p be the Probability of any event; a gambler wagers, with a casts permitted to bring forth an event in a certain number of trials which surpasses by b units the number of trials, in which the event does not appear. The lot of the gambler is sought.

Solution.

This problem and which precede are solved in a not dissimilar manner. The sought Probability is the sum of the following Probabilities:

1) With b casts permitted, the event to arrive with b trials, of which case the Probability is p^b.

2) With $b+2$ casts permitted, the event to arrive with $b+1$ trials only, of which case the Probability is $p^{b+1}(1-p) = p^{b+1}q$ (by supposing $q = 1-p$). This term is multiplied by the coefficient, which we will seek out in the following way. The number of permutations of $b+2$ quantities, of which $b+1$ are the same is $= b+2$. The cases should be subtracted, in which with b casts the game is completed. But with the event p brought forth b times first, there remain two trials in which p and q are able to arrive in two ways, therefore the coefficient is $= b+2-2 = b$, and the Probability of this case is $= bp^{b+1}q$. (I have omitted the supposition of $b+1$ trials, as impossible, for if q units at least arrive by chance, the game is not able to be completed with $b+1$ casts; if it not arrive, the game is finished with b casts. For the same reason I will omit the suppositions of $b+3$, $b+5$, $b+7$ etc. casts.)

3. With $b+4$ casts permitted, the event to arrive with $b+2$ trials only, of which case the Probability is the term $p^{b+2}q^2$ multiplied by the coefficient which now I will seek

[7] *Translator's note*. Trembley refers here to the 2nd edition. This problem was incorporated into Problem VI of the 3rd.
[8] *Translator's note*. This is a Problem of the Duration of Play. See Problem LXV of Moivre.

out. The number of permutations of $b+4$ quantities, of which $b+2$ are the same and 2 the same is $\frac{(b+4)(b+3)}{2}$. The cases should be subtracted, in which the game is completed with $b+2$ casts, and the case in which the game is completed with b casts. In order that the game may be completed in $b+2$ casts, q must arrive once only in the first b, therefore there remain in the end p and q of which the permutations are 2; but through the preceding case b combinations arrive in which the game is finished with $b+2$ casts, therefore $2b$ cases should be subtracted. In order that the game may be completed with b casts, q must not appear in the first b, therefore there remain in the end $2p$ and $2q$, of which the combinations are $= \frac{1.2.3.4}{1.2.1.2} = 6$. Therefore there should be subtracted in all $2b+6 = \frac{4(b+3)}{2}$ cases. The coefficient will be $\frac{(b+4)(b+3)}{2} - \frac{4(b+3)}{2} = \frac{b(b+3)}{2}$ and the Probability of this case is $= \frac{b(b+3)}{2} p^{b+2} q^2$.

4) With $b+6$ casts permitted, the event to arrive with $b+3$ trials only, of which case the Probability is the term $p^{b+3}q^3$ multiplied by the coefficient which now I will seek. The number of permutations of $b+6$ quantities, of which $b+3$ are the same, and 3 the same is $= \frac{(b+6)(b+5)(b+4)}{1.2.3}$. The cases are subtracted in which the game is completed with $b+4$, $b+2$, b trials. In order that the game may be completed in $b+4$ trials, in this q must appear twice, therefore there remains in the end p and q of which the permutations are $= 2$, but by the previous case, $\frac{b(b+3)}{2}$ combinations arrive, in which the game is completed with $b+4$ casts, therefore $b(b+3)$ cases should be subtracted. In order that the game may be completed with $b+2$ casts, in this q must appear once, there remain therefore in the end $2p$ and $2q$ of which the permutations are $= 6$, and by the second case, the number of combinations, in which the game is finished with $b+2$ casts is $= b$, therefore $6b$ cases should be subtracted. In order that the game may be completed with b casts, q must not appear in the first b, therefore there remain in the end $3p$ and $3q$ of which the permutations are $\frac{1.2.3...6}{1.2.3.1.2.3} = 20$. Therefore 20 cases should be subtracted therefore there should be subtracted in total

$$b(b+3) + 6b + 20 = (b+4)(b+5) = \frac{6(b+4)(b+5)}{2.3}$$

cases. Therefore the coefficients will be

$$= \frac{(b+6)(b+5)(b+4)}{2.3} - \frac{6(b+5)(b+4)}{2.3} = \frac{b(b+4)(b+5)}{2.3}$$

and the Probability of this case will be

$$= \frac{b(b+4)(b+5)}{2.3} p^{b+3} q^3.$$

5) With $b+8$ casts permitted, the event to arrive with $b+4$ trials only, of which case the Probability is the term $p^{b+4}q^4$ multiplied by the coefficient which we will seek. The number of Permutations of $b+8$ quantities of which $b+4$ are the same and 4 are the same is $= \frac{(b+8)(b+7)(b+6)(b+5)}{1.2.3.4}$. The cases should be subtracted in which the game is completed in a smaller number of casts. In order that the game may be completed in $b+6$ casts, q must arrive three times in this, therefore there remain in the end p and q of which the permutations are $= 2$; but by the preceding case, there arrive $\frac{b(b+4)(b+5)}{2.3}$ of which the game is completed in $b+6$ casts. Therefore the first number should be

subtracted $= \frac{b(b+4)(b+5)}{3}$. In order that the game may be terminated with $b+4$ casts q must arrive twice in this, therefore there remains $2p$ and $2q$ of which the permutations are $= 6$, and by the third case there arrive $\frac{b(b+3)}{2}$ combinations of which the game will be terminated with $b+4$ casts. Therefore the second number should be subtracted $= 3b(b+3)$. In order that the game may be terminated in $b+2$ casts, q must be in this one time, therefore there remain $3p$ and $3q$ of which the permutations are 20, but by the second case there arrive b combinations in which the game will be terminated in b casts. Therefore the third number should be subtracted $= 20b$. Finally in order that the game may be terminated in b casts, q must not appear in this, therefore there remain $4p$ and $4q$ of which the permutations are $= 70$. Therefore the fourth number should be subtracted $= 70$. Therefore there should be subtracted in all cases

$$\begin{aligned} &= \frac{b(b+4)(b+5)}{3} + \frac{3b(b+3)}{2} + 20b + 70 \\ &= \frac{(b+7)(b+6)(b+5)}{3} \\ &= \frac{8(b+7)(b+6)(b+5)}{2.3.4}. \end{aligned}$$

Therefore the coefficient will be

$$\begin{aligned} &= \frac{(b+8)(b+7)(b+6)(b+5)}{1.2.3.4} - \frac{8(b+7)(b+6)(b+5)}{1.2.3.4} \\ &= \frac{b(b+7)(b+6)(b+5)}{1.2.3.4}, \end{aligned}$$

and the Probability of this case will be

$$= \frac{b(b+5)(b+6)(b+7)}{1.2.3.4} p^{b+4} q^4.$$

6) With $b+10$ casts permitted, the event to arrive $b+5$ times only, of which case the Probability is the term $p^{b+5}q^5$ multiplied by the coefficient, which we will seek. The number of Permutations of $b+10$ quantities, of which $b+5$ are the same and 5 the same is $= \frac{(b+6)\ldots(b+10)}{1.2\ldots5}$. The cases should be subtracted in which the game is terminated in a fewer number of casts. But it is clear from the preceding, with the same method used,
$\frac{b(b+5)(b+6)(b+7)}{3.4}$ cases to arrive in which the game is terminated in $b+8$ casts,
$b(b+4)(b+5)$ cases in which the game is terminated in $b+6$ casts,
$10b(b+3)$ cases in which the game is terminated in $b+4$ casts,
$70b$ cases in which the game is terminated in $b+2$ casts,
252 cases in which the game is terminated in b casts.
The sum of these terms is

$$= \frac{10(b+6)(b+7)(b+8)(b+9)}{1.2.3.4.5}.$$

Therefore the coefficient will be

$$= \frac{b(b+6)(b+7)(b+8)(b+9)}{1.2.3.4.5}$$

and the Probability of this case will be

$$= \frac{b(b+6)(b+7)(b+8)(b+9)}{1.2.3.4.5} p^{b+5} q^5.$$

Now the law of progression is evident, from which it is clear, the Probability with $b+2n$ casts permitted the event to arrive with $b+n$ coups and to be lacking by n coups to be

$$= \frac{b(b+n-1)(b+n-3)\dots(b+2n-1)}{1.2.3\dots n} p^{b+n} q^n.$$

Therefore the lot of the gambler will be

$$\begin{aligned} = p^b \Bigg(1 + bpq + \frac{b(b+3)}{2} p^2 q^2 + \frac{b(b+4)(b+5)}{2.3} p^3 q^3 \\ + \frac{b(b+5)(b+6)(b+7)}{2.3.4} p^4 q^4 + \frac{b(b+6)(b+7)(b+8)(b+9)}{2.3.4.5} p^5 q^5 \dots \\ + \frac{b(b+n-1)\dots(b+2n-1)}{1.2\dots n} p^n q^n \Bigg). \end{aligned}$$

But the number of casts is not able to exceed a, whence $b+2n=a$, $n=\frac{a-b}{2}$, therefore the number of terms must be assumed $= \frac{a-b}{2}+1$ if $a-b$ is an even number, and $= \frac{a-b-1}{2}+1$ if it is odd. The same consequences result from the solution of the Celebrated Lagrange.

Scholium.

10. This Problem coincides with Problem LXIV[9] of Moivre, but our solution coincides with the second solution of the most celebrated Author. We solve examples here which Moivre reports. Let there be two gamblers A and B, of whom the Probabilities are respectively a and b, with each cast the defeated gives a coin to the victor, the Probability is sought of A to win 3 coins within 10 casts. Here there is $p = \frac{a}{a+b}$, $q = \frac{b}{a+b}$, $a = 10$, $b = 3$, the number of terms of the formula must be $= \frac{6}{2}+1 = 4$. Therefore the formula will become

$$\frac{a^3}{(a+b)^3}\left(1 + \frac{3ab}{(a+b)^2} + \frac{9a^2b^2}{(a+b)^4} + \frac{28a^3b^3}{(a+b)^6}\right) = \frac{a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 36a^5b^4 + 9a^4b^5 + a^3b^6}{(a+b)^9}$$

as Moivre reports. It is sought again, under the same assumptions, A winning 4 coins within 10 casts, here there is $b = 4$, with all the rest remaining. Therefore the formula will become

$$\frac{a^4}{(a+b)^4}\left(1 + \frac{4ab}{(a+b)^2} + \frac{12a^2b^2}{(a+b)^4} + \frac{48a^3b^3}{(a+b)^6}\right) = \frac{a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 45a^6b^4 + 10a^5b^5 + a^4b^6}{(a+b)^{10}}$$

[9]*Translator's note*. In the 3rd edition, this is Problem LXV.

as Moivre reports.

Problem 7.[10]

11. Let as in the preceding Problem p be the Probability of any event, the gambler wagers, with a casts permitted, to bring forth the event in a certain number of trials, which exceed by b units the number of trials, in which the event will not arrive, or be inferior by c units to the same number of trials, in which namely the event will not arrive.

Solution.

It is evident the sought Probability to correspond to the following Probabilities: 1) The number of cases in which the event will arrive surpassing by b units the number of cases in which it will not arrive. 2) The number of cases in which the event not arrive surpassing by c units the number of cases in which it will arrive. But certain cases are contained twice in these formulas, and hence the cases should be subtracted from the Probability a priori, in which before the end of the game the number of casts in which the event has not arrived surpasses by c units the number of casts in which it has arrive, for that number of cases already is contained in the second Probability, in the same way, a posteriori the cases should be subtracted from the Probability in which the number of casts in which the event has arrived surpasses by b units the number of cases in which it has not arrived, for that number already is contained in the first Probability.

This posed, by the preceding Problem the Probability of the event arriving within a casts in a certain number of trials which surpass by b units the number of trials in which the event has not arrived, is

$$p^b\left(1+bpq+\frac{b(b+3)}{2}p^2q^2+\frac{b(b+4)(b+5)}{2.3}p^3q^3\dots\right.$$
$$\left.+\ \frac{b(b+1.2+1)\dots(b+2n-1)}{1.2\dots n}p^nq^n\right)$$

(here $b+n=a$). Let now $n=c+d$, the preceding formula will become

$$p^b\left(1+bpq+\frac{b(b+3)}{2}p^2q^2+\frac{b(b+4)(b+5)}{2.3}p^3q^3\dots\right.$$
$$+\ \frac{b(b+c+1)\dots(b+2c-1)}{1.2\dots c}p^cq^c$$
$$+\ \frac{b(b+c+2)\dots(b+2c+1)}{1.2\dots(c+1)}p^{c+1}q^{c+1}$$
$$+\ \frac{b(b+c+3)\dots(b+2c+3)}{1.2\dots c}p^{c+2}q^{c+2}$$
$$\left.+\ \frac{b(b+c+d+1)\dots(b+2c+2d-1)}{1.2\dots(c+d)}p^{c+d}q^{c+d}\right)$$

[10]*Translator's note*. This is Problem VI of Lagrange. See § 61.

It is plain from this formula, as long as it considers the terms which precede the term $p^c q^c$, the number of cases in which the event will not arrive to be impossible, to surpass by c trials the number of cases in which it will arrive. For the term $p^c q^c$ the fact is possible in one way, thus as for this case the term $p^{b+c}q^c$ should be subtracted. For the term $p^{b+c+1}q^{c+1}$, in first place they produce the cases in which q^c precedes, with q and p^{b+c+1} undergoing all permutations, of which the number is $b+c+2$. Then they produce the cases in which q^{c+1} and p are combined, with the remaining terms arising $= p^{b+c}$, the permutations of these cases are $= c+2$, moreover the cases contained in the preceding should be subtracted, certainly $q^c.\,qp$, with qp undergoing all permutations of which the number is$=2$, therefore there remains $c+2-2=c$, we have therefore $b+2c+2$ but the cases should be subtracted in which the game will be terminated with $b+2c$ casts, it is the case of this formula $p^{b+c}q^c.qp$, with qp undergoing all permutations of which the number$=2$, therefore there remains $b+2c$. In this case therefore the term $(b+2c)p^{b+c+1}q^{c+1}$ should be subtracted.

For the term $p^{b+c+2}q^{c+2}$ the same will be obtained by the distinction of the cases employed which applies to the place in the preceding Problem, the following table,

1) The substitution $q^c.q^2p^{b+c+2}$ of which the cases are $\frac{(b+c+3)(b+c+4)}{1.2}$.

2) The substitution $q^{c+1}p.p^{b+c+1}$ of which the cases are $c(b+c+2)$.

2) The substitution $q^{c+2}p^2.p^{b+c}$ of which the cases are $\frac{c(c+3)}{1.2}$.

The sum of these cases is $\frac{(b+c+3)(b+c+4)}{1.2}+c(b+c+2)+\frac{c(c+3)}{1.2}$. The cases should be subtracted in which the game will be terminated in a fewer number of casts, which by the method used above are discovered $=(b+2c)-6$. Therefore the coefficient is

$$\frac{(b+c+3)(b+c+4)}{1.2}+c(b+c+2)+\frac{c(c+3)}{1.2}-2(b+2c)+6$$
$$=\frac{(b+2c+3)(b+2c)}{2},$$

and the term to be subtracted $\frac{(b+2c+3)(b+2c)}{2}p^{b+c+2}q^{c+2}$.

For the term $p^{b+c+3}q^{c+3}$, likewise there appear

1) $q^c.q^3p^{b+c+3}$ of which substitution the cases are $\frac{(b+c+4)(b+c+5)(b+c+6)}{2.3}$,

2) $q^{c+1}p.q^2p^{b+c+2}$ of which substitution the cases are $\frac{c(b+c+3)(b+c+4)}{2}$,

3) $q^{c+2}p^2.qp^{b+c+1}$ of which substitution the cases are $\frac{c(c+3)}{2}(b+c+2)$,

4) $q^{c+3}p^3.p^{b+c}$ of which substitution the cases are $\frac{c(c+4)(c+5)}{2.3}$.

The sum of these cases is

$$\frac{(b+c+4)(b+c+5)(b+c+6)}{2.3}+\frac{c(b+c+3)(b+c+4)}{2}$$
$$+\frac{c(c+3)}{2}(b+c+2)+\frac{c(c+4)(c+5)}{2.3}.$$

The cases should be subtracted in which the game will be terminated in a smaller number of casts which are discovered $= \frac{2(b+2c+3(b+2c)}{2} + 6(b+2c) + 20$. Therefore the coefficient after reductions is

$$= \frac{(b+2c)(b+2c+4)(b+2c+5)}{1.2.3}.$$

and the term

$$= \frac{(b+2c)(b+2c+4)(b+2c+5)}{1.2.3} p^{b+c+3} q^{c+3}.$$

should be subtracted.

Now the law of progression is clear, and the general formula of the terms to be subtracted will be

$$p^{b+c} q^c \left(1 + (b+2c)pq + \frac{(b+2c)(b+2c+3)}{2} p^2 q^2 \right.$$
$$+ \frac{(b+2c)(b+2c+4)(b+2c+5)}{2.3} p^3 q^3 \ldots$$
$$\left. + \frac{(b+2c)(b+2c+d+1)\ldots+(b+2c+2d-1)}{1.2\ldots d} p^d q^d \right).$$

Now let there be $d = b + m$, and the formula will become

$$p^{b+c} q^c \left(1 + (b+2c)pq + \frac{(b+2c)(b+2c+3)}{2} p^2 q^2 \ \ldots \right.$$
$$+ \frac{(b+2c)(b+2c+1)(3b+2c-1)}{2.3\ldots b} p^b q^b \ldots$$
$$+ \frac{(b+2c)(2b+2c+2)(3b+2c+1)}{2.3\ldots(b+1)} p^{b+1} q^{b+1}$$
$$\left. + \frac{(b+2c)(b+2c+m+1)\ldots(3b+2c+2m-1)}{2.3\ldots(b+m)} p^{b+m} q^{b+m} \right).$$

This second series permits the same exceptions as the first. Because it observes the terms which precede the term $p^b q^b$, it is impossible the number of cases in which the event will arrive to exceed by b trials the number of cases in which it not appear, from this term the thing is possible, and the terms to be subtracted are discovered in precisely the same manner as above, thus as the third series which must be added, when it should be subtracted from the series it must be subtracted

$$p^{2b+c} q^{b+c} \left(1 + (3b+2c)pq + \frac{(3b+2c)(3b+2c+3)}{2} p^2 q^2 \right.$$
$$\left. + \frac{(3b+2c)(3b+2c+m+1)\ldots(3b+2c+2m-1)}{1.2.3\ldots m} p^m q^m \right).$$

The same reasoning is able to proceed to infinity. But we have $n = \frac{a-b}{2}$ or $n = \frac{a-b-1}{2}$, I will use the first value, it produces $d = n - c = \frac{a-b-2c}{2}$, $m = d - b = \frac{a-3b-2c}{2}$, and thus successively. Now this formula exists generally,

$$p^b\Bigg(1+bpq+\tfrac{b(b+3)}{2}p^2q^2\ldots+\tfrac{b\left(\frac{a+b}{2}+1\right)\left(\frac{a+b}{2}+2\right)\ldots(a-1)}{1.2.3}p^{\frac{a-b}{2}}q^{\frac{a-b}{2}}$$

$$-p^{b+c}q^c\Big(1+(b+2c)pq+\tfrac{(b+2c)(b+2c+3)}{1.2}p^2q^2$$

$$+\tfrac{(b+2c)(b+2c+4)(b+2c+5)}{2.3}p^3q^3\ldots$$

$$+\tfrac{(b+2c)\left(\frac{a+b+2c}{2}+1\right)\left(\frac{a+b+2c}{2}+2\right)\ldots(a-1)}{1.2.3\ldots\left(\frac{a+b+2c}{2}\right)}p^{\frac{a-b-2c}{2}}q^{\frac{a-b-2c}{2}}\Bigg)$$

$$+p^{2b+c}q^{b+c}\Big(1+(3b+2c)pq+\tfrac{(3b+2c)(3b+2c+3)}{2}p^2q^2\ldots$$

$$+\tfrac{(3b+2c)\left(\frac{a+3b+2c}{2}+1\right)\left(\frac{a+3b+2c}{2}+2\right)\ldots(a-1)}{1.2.3\ldots\left(\frac{a+3b+2c}{2}\right)}p^{\frac{a-3b-2c}{2}}q^{\frac{a-3b-2c}{2}}$$

$$-p^{2b+2c}q^{b+2c}\Big(1+(3b+4c)pq+\tfrac{(3b+4c)(3b+4c+3)}{2}p^2q^2\ldots$$

$$+\tfrac{(3b+4c)\left(\frac{a+3b+4c}{2}+1\right)\left(\frac{a+3b+4c}{2}+2\right)\ldots(a-1)}{1.2.3\ldots\left(\frac{a+3b+4c}{2}\right)}p^{\frac{a-3b-4c}{2}}q^{\frac{a-3b-4c}{2}}\Bigg)$$

$$+p^{3b+2c}q^{2b+2c}\Big(1+(5b+4c)pq+\tfrac{(5b+4c+3)}{2}p^2q^2\ldots$$

$$-\tfrac{(5b+4c)\left(\frac{a+5b+4c}{2}+1\right)\left(\frac{a+5b+4c}{2}+2\right)\ldots(a-1)}{1.2.3\ldots\left(\frac{a+5b+4c}{2}\right)}p^{\frac{a-5b-4c}{2}}q^{\frac{a-5b-4c}{2}}\Bigg)$$

$$-p^{\mu b+\mu c}q^{(\mu-1)b+\mu c}\left[1+[(2\mu-1)b+2\mu c]pq\right.$$

$$+\tfrac{[(2\mu-1)b+2\mu c][(2\mu-1)b+2\mu c+3]}{2}p^2q^2\ldots$$

$$+\tfrac{[(2\mu-1)b+2\mu c]\left(\frac{a+(2\mu-1)b+2\mu c}{2}+1\right)\left(\frac{a+(2\mu-1)b+2\mu c}{2}+2\right)\ldots(a-1)}{1.2.3\ldots\left(\frac{a+(2\mu-1)b+2\mu c}{2}\right)}$$

$$\left.p^{\frac{a+(2\mu-1)b+2\mu c}{2}}q^{\frac{a+(2\mu-1)b+2\mu c}{2}}\right]$$

$$+p^{(\mu+1)b+\mu c}q^{\mu b+\mu c}\left[1+[(2\mu+1)b+2\mu c]pq\right.$$

$$+\tfrac{[(2\mu+1)b+2\mu c][(2\mu+1)b+2\mu c+3]}{1.2}p^2q^2\ldots$$

$$+\tfrac{[(2\mu+1)b+2\mu c]\left(\frac{a+(2\mu+1)b+2\mu c}{2}+1\right)\left(\frac{a+(2\mu+1)b+2\mu c}{2}+2\right)\ldots(a-1)}{1.2.3\ldots\left(\frac{a-(2\mu+1)b-2\mu c}{2}\right)}$$

$$\left.p^{\frac{a-(2\mu-1)b-2\mu c}{2}}q^{\frac{a-(2\mu-1)b-2\mu c}{2}}\right]$$

The last two formulas express the $2\mu^{\text{th}}$ and $(2\mu+1)^{\text{th}}$ formula, and hence the general formula is able to be continued to infinity.

This formula contains the first part of the expectation of the gambler, namely the Probability of the numbers of cases, in which the event will happen being about to surpass by b units the number of cases, in which it is not contained, with the Probability excluded before this time of the numbers of cases, in which the event not happen being about to surpass by c units the number of cases, in which it happens.

The second part of the expectation of the gambler expresses the Probability of the numbers of cases, in which the event not happen being about to surpass by c units the number of cases in which it happens, with the Probability excluded before this time the number of cases, in which the event not happen being about to exceed by b units the number of cases in which it not happen, but it is obtained by substituting into the first q for p and p for q, likewise b for c and c for b. Therefore the formula will become

$$q^c\left[1+cpq+\tfrac{c(c+3)}{2}p^2q^2\ldots+\tfrac{c\left(\frac{a+c}{2}+1\right)\left(\frac{a+c}{2}+2\right)\ldots(a-1)}{1.2.3}p^{\frac{a-c}{2}}q^{\frac{a-c}{2}}\right]$$

$$-q^{b+c}p^b\Big[1+(c+2b)pq+\tfrac{(c+2b)(c+2b+3)}{1.2}p^2q^2$$

$$
\begin{aligned}
&+\tfrac{(c+2b)(c+2b+4)(c+2b+5)}{2.3}p^3q^3\dots\\
&+\tfrac{(c+2b)\left(\frac{a+c+2b}{2}+1\right)\left(\frac{a+c+2b}{2}+2\right)\dots(a-1)}{1.2.3\dots\left(\frac{a-c-2b}{2}\right)}p^{\frac{a-c-2b}{2}}q^{\frac{a-c-2b}{2}}\Big]\\
&+q^{2c+bc}p^{c+b}\Big[1+(3c+2b)pq+\tfrac{(3c+2b)(3c+2b+3)}{2}p^2q^2\dots\\
&+\tfrac{(3c+2b)\left(\frac{a+3c+2b}{2}+1\right)\left(\frac{a+3c+2b}{2}+2\right)\dots(a-1)}{1.2.3\dots\left(\frac{a+3c+2b}{2}\right)}p^{\frac{a-3c-2b}{2}}q^{\frac{a-3c-2b}{2}}\Big]\\
&-q^{2b+2c}p^{c+2b}\Big(1+(3c+4b)pq+\tfrac{(3c+4b)(3c+4b+3)}{2}p^2q^2\dots\\
&+\tfrac{(3c+4b)\left(\frac{a+3c+4b}{2}+1\right)\left(\frac{a+3c+4b}{2}+2\right)\dots(a-1)}{1.2.3\dots\left(\frac{a+3c+4b}{2}\right)}p^{\frac{a-3c-4b}{2}}q^{\frac{a-3c-4b}{2}}\Big]\\
&+q^{3c+2b}p^{2c+2b}\Big[1+(5c+4b)pq+\tfrac{(5c+4b)(5c+4b+3)}{2}p^2q^2\dots\\
&-\tfrac{(5c+4b)\left(\frac{a+5c+4b}{2}+1\right)\left(\frac{a+5c+4b}{2}+2\right)\dots(a-1)}{1.2.3\dots\left(\frac{a+5b+4c}{2}\right)}p^{\frac{a-5c-4b}{2}}q^{\frac{a-5c-4b}{2}}\Big]\\
&-q^{\mu b+\mu c}p^{(\mu-1)c+\mu b}\left[1+[(2\mu-1)c+2\mu b]pq\right.\\
&+\tfrac{[(2\mu-1)b+2\mu b][(2\mu-1)c+2\mu b+3]}{2}p^2q^2\dots\\
&+\tfrac{[(2\mu-1)c+2\mu b]\left(\frac{a+(2\mu-1)c+2\mu b}{2}+1\right)\left(\frac{a+(2\mu-1)c+2\mu b}{2}+2\right)\dots(a-1)}{1.2.3\dots\left(\frac{a-(2\mu-1)c-2\mu b}{2}\right)}\\
&p^{\frac{a+(2\mu-1)c+2\mu b}{2}}q^{\frac{a+(2\mu-1)c+2\mu b}{2}}\Big]\\
&+q^{(\mu+1)c+\mu b}q^{\mu c+\mu b}\left[1+[(2\mu+1)c+2\mu b]pq\right.\\
&+\tfrac{[(2\mu+1)c+2\mu b][(2\mu+1)c+2\mu b+3]}{1.2}p^2q^2\dots\\
&+\tfrac{[(2\mu+1)c+2\mu b]\left(\frac{a+(2\mu+1)c+2\mu b}{2}+1\right)\left(\frac{a+(2\mu+1)c+2\mu b}{2}+2\right)\dots(a-1)}{1.2.3\dots\left(\frac{a-(2\mu+1)c-2\mu b}{2}\right)}\\
&p^{\frac{a-(2\mu-1)c-2\mu b}{2}}q^{\frac{a-(2\mu-1)c-2\mu b}{2}}\Big]
\end{aligned}
$$

From these two joined formulas the expectation of the gambler is understood. If $a-b$ be an odd number, there must be $n=\frac{a-b-1}{2}$, therefore $d=n-c=\frac{a-b-1-2c}{2}$, $n=d-b=\frac{a-3b-1-3c}{2}$, and thus successively. But it is not useful that our formulas be written anew, it suffices in the case of the odd number $a-b$ to use the nearest inferior even number. The solution reverts to the same of the Celebrated Lagrange, although the same formulas are not produced. We solve the same example which the Celebrated man proposes. In this example there happens $a=7$, $b=2$, $c=3$. First the first part of our formulas will become $p^2(1+2pq+\frac{2.5}{2}p^2q^2)$, I stop here, because $\frac{a-b}{2}=\frac{5}{2}=2$, the second part, and all the others vanish, because $\frac{a-b-2c}{2}=-\frac{1}{2}$. The first part of the second formula becomes $q^3(1+3pq+\frac{3.6}{2}p^2q^2)$, I stop here because $\frac{a-c}{2}=2$. The second part, and all the others vanish, because $\frac{a-c-2b}{2}=0$. Therefore the expectation of the gambler appears

$$
\begin{aligned}
&=p^2+2p^3q+5p^4q^2+q^3+3pq^4+8p^2q^5\\
&=p^2+2p^3(1-p)+5p^4(1-p)^2+(1-p)^3+3p(1-p)^4+8p^2(1-p)^5\\
&=1-21p^3+71p^4-87p^5+45p^6-8p^7
\end{aligned}
$$

as the Celebrated Lagrange reports.

Scholium 1.

12. Here Problem 62[11] of Moivre returns, in which namely the complement of the Probability is sought which we sought here. We have been seeking the Probability of the games being terminated within a certain number of casts, but Moivre the Probability of the games not being terminated. For Problem 63[12] of Moivre is the same kind. In the example reported there is $a = 15$, $b = 2$, $c = 3$. Therefore the first formula will become

$$\begin{aligned}&p^2\left(1+2pq+\frac{2.5}{2}p^2q^2+\frac{2.6.7}{2.3}p^3q^3+\frac{2.7.8.9}{2.3.4}p^4q^4\right.\\&\quad\left.+\frac{2.8.9.10.11}{2.3.4.5}p^5q^5+\frac{2.9.10.11.12.13}{2.3.4.5.6}p^6q^6\right)\\&\quad-p^5q^3\left(1+8pq+\frac{8.11}{2}p^2q^2+\frac{8.12.13}{2.3}p^3q^3\right)+p^7q^5(1+12pq)\\&=p^2+2p^3q+5p^4q^2+13p^5q^3+34p^6q^4+89p^7q^5+233p^8q^6.\end{aligned}$$

The second formula will become

$$\begin{aligned}&q^3\left(1+3pq+\frac{3.6}{2}p^2q^2+\frac{3.7.8}{2.3}p^3q^3+\frac{3.8.9.10}{2.3.4}p^4q^4\right.\\&\quad\left.+\frac{3.9.10.11.12}{2.3.4.5}p^5q^5+\frac{3.10.11.12.13.14}{2.3.4.5.6}p^6q^6\right)\\&\quad-p^2q^5\left(1+7pq+\frac{7.10}{1.2}p^2q^2+\frac{7.11.12}{1.2.3}p^3q^3+\frac{7.12.13.14}{1.2.3.4}p^4q^4\right)\\&\quad+p^5q^8(1+13pq)\\&=q^3+3pq^4+8p^2q^5+21p^3q^6+55p^4q^7+143p^5q^8+377p^6q^9.\end{aligned}$$

The expectation of the gambler therefore will become=

$$\begin{aligned}p^2&+2p^3q+5p^4q^2+13p^5q^3+34p^6q^4+89p^7q^5+233p^8q^6\\&+q^3+3pq^4+8p^2q^5+21p^3q^6+55p^4q^7+143p^5q^8+377p^6q^9.\end{aligned}$$

Which formula coincides with the formula of Moivre if there are made $p = \frac{a}{a+b}$, $q = \frac{b}{a+b}$.

Scholium 2.

13. If with close attention the two formulas are assessed carefully, of which the last presents the general solution of our problem, it will be well-known the first part of the first formula, of which the factor is p^b, to be nothing other, if not the formula of Problem 6, namely to express the Probability A to win b coins however many the number of coins B should have gained; but the other part of this first formula, of which the factors

[11] *Translator's note*. In the 3rd edition, this is Problem LXIII.
[12] *Translator's note*. In the 3rd edition, this is Problem LXIV. The particular case is on p. 205.

are respectively $p^{b+c}q^{c}$, $p^{b+2c}q^{b+c}$, $p^{2b+2c}q^{b+2c}$ etc. to express the Probability A to win b coins within a casts before B may have won anything. In the same manner the first part of the second formula, of which the factor is q^c, is nothing other, if not the Prob. 6 formula if the Probability A to be losing b coins is sought, it provides the Probability B to win c coins, however many the number of coins A may have gained; but the other parts of this second formula, of which the factors are $q^{b+c}p^{b}$, $q^{2c+b}p^{b+c}$ etc. respectively present the Probability B to win c coins, before A may have won any. Therefore the sum of the Probabilities of these latter parts present the Probability, within a casts A to win b coins, but B c coins. Therefore with the first part of each formula removed, our Problem will give the solution to Prob. 65[13] of Moivre without any runaround. For the sake of an example, the Probability is sought within 10 casts A to win 2 coins, also B three. This will become $a = 10$, $b = 2$, $c = 3$, and the second part of the first formula will present $p^5q^3(1 + 8pq)$. I stop here, because $\frac{a-b-2c}{2} = \frac{10-8}{2} = 1$, moreover the other parts vanish, because $\frac{a-3b-2c}{2} = -1$. The second part of the second formula presents $p^2q^5(1+7pq)$. I stop here, because $\frac{a-c-2b}{2} = \frac{10-7}{2} = \frac{3}{2} = 1$, moreover the other parts vanish, because $\frac{a-3c-2b}{2} = \frac{10-11}{2} = -\frac{1}{2}$. Therefore the sought Probability will be $p^5q^3+8p^6q^4+p^2q^5+7p^3q^6$, as Moivre reports. If in addition the Probability A to win 3 coins and B 2 is sought, we will obtain by permuting p and q in the discovered terms,

$$p^3q^5 + 8p^4q^6 + p^5q^2 + 7p^6q^3,$$

moreover the entire Probability will be

$$(pp + qq)(p^3q^3 + 8p^4q^4) + (p^3 + q^3)(p^2q^2 + 7p^3q^3)$$

as Moivre reports. In order that a comparison may be made, it must be supposed $p = \frac{a}{a+b}, q = \frac{b}{a+b}$ as above.

Scholium 3.

14. By aid of this Problem Problem 66[14] of Moivre is solved besides in which the Probability is sought within a casts A to win q coins, but B not to win p coins, in fact in the second formula there must happen $c = p - 1$.

Problem 8.[15]

15. With any one cast two events are able to happen, of which the Probabilities are p and q, the lot of the gambler is sought, who wagers to bring forth the first event b times at least, and the second event c times at least within a casts.

[13] *Translator's note*. In the 3rd edition, this is Problem LXVI.
[14] *Translator's note*. In the 3rd edition, this is Problem LXVII.
[15] *Translator's note*. This is Problem II of Lagrange.

Solution.

We have solved an analogous question in the second Problem, but there one event arrived, of which the Probability was p, and generally we have exhibited the lot of the gambler for this case. The same method used for the two events may offer lengthier formulas, it is permitted in certain cases they may be abbreviated. Therefore in order that we may obtain the general solution of the Problem, we suppose known through Prob. 2 the lot of the gambler in the case of one event, and we drive from the other event, of which the Probability is q, just as we have driven from the first in the Problem cited. The Probability of bringing forth this event c times by following themselves is q^c, this Probability should be multiplied by the lot of the gambler, when he wagers to bring forth the event p b times within $a-c$ casts. Let therefore by using the denominations of the Celebrated Lagrange, $7_{x,t}$ be the lot of the gambler, who wagers to bring forth a certain event t times at least, within x casts, the Probability of the case, in which the event q happens in the first c trials, will be $q^c 7_{a-c,\,b}$.

We come to the second case in which the event q happens in c trials within the first $c+1$ casts; the number of cases under this assumption is as we proved in Prob. 2, $= c$. Therefore in this case when the event q may arrive c times within $c+1$ casts, in the cast which is in excess the event p will arrive or not. In the first case the Probability will be the term $cq^c p$ multiplied by the lot of the gambler, who wagers to bring forth the event p, with $b-1$ trials within $a-c-1$ casts, namely $= cq^c p 7_{a-c-1,\,b-1}$. In the second case the Probability will be the term $cq^c(1-p-q) = cq^c n$ (by supposing $n = 1-p-q$) multiplied by the lot of the gambler, who wagers to bring forth the event p, b times within $a-c-1$ casts, namely $= cq^c n 7_{a-c-1,\,b}$. The Probability of this second case will be therefore $= cq^c(n7_{a-c-1,\,b} + p7_{a-c-1,\,b-1})$.

The third case is this in which the event q happens c times within $c+2$ casts, the number of cases under this assumption is through Prob. 2, $= \frac{c(c+1)}{2}$. Therefore in this case when the event q may arrive c times within $c+2$ casts, in the remaining two casts there will arrive either p^2 or n^3 or $2np$ as it is clear from Prob. 3. In the first case the Probability will be the term $\frac{c(c+1)}{2} q^c p^2$ multiplied by the lot of the gambler, who wagers to bring forth the event p, $b-2$ times within $a-c-2$ casts, namely $= \frac{c(c+1)}{2} q^c p^2 7_{a-c-2,\,b-2}$. In the second case the Probability will be the term $\frac{c(c+1)}{2} q^c n^2$, multiplied by the lot of the gambler, who wagers to bring forth the event p, b times within $a-c-2$ casts, namely $= \frac{c(c+1)}{2} q^c n^2 7_{a-c-2,\,b}$. In the third case the Probability will be the term $\frac{c(c+1)}{2} q^c.2np$ multiplied by the lot of the gambler, who wagers to bring forth the event p, $b-1$ times within $a-c-2$ casts, namely $= \frac{c(c+1)}{2} q^c.2np7_{a-c-2,\,b-1}$. The Probability of this third case will be therefore $=$

$$\frac{c(c+1)}{2} q^c(n^2 7_{a-c-2,\,b} + 2npq7_{a-c-2,\,b-a} + p^2 7_{a-c-2,\,b-c}).$$

The fourth case is this in which the event q happens c times within $c+3$ casts. The number of cases under this supposition is through Prob. 2, $= \frac{c(c+1)(c+2)}{2.3}$. Therefore in this case, when the event q arrives c times within $c+3$ casts, in the remaining three casts there arrive either p^3 or $3p^2 n$ or $3pn^2$ or n^3, as it is clear from Prob. 4. In the first case the Probability will be the term $\frac{c(c+1)(c+2)}{2.3}$ multiplied by the lot of the gambler, who wagers to bring forth the event p, with $b-3$ trials within $a-c-3$ casts, namely $= \frac{c(c+1)(c+2)}{2.3} q^c p^3 7_{a-c-3,\,b-3}$. In the second case the Probability will be the term $\frac{c(c+1)(c+2)}{2.3} q^c.3p^2 n$ multiplied by the lot of the gambler, who wagers to bring forth the event p, by $b-2$ trials within $a-c-3$ casts, namely $= \frac{c(c+1)(c+2)}{2.3} q^c.3p^2 n 7_{a-c-3,\,b-2}$. In the third case the Probability will be the term $\frac{c(c+1)(c+2)}{2.3} q^c 3n^2 p$ multiplied by the lot of the gambler who wagers to bring forth, the event p, by $b-1$ trials within $a-c-3$ casts, namely $= \frac{c(c+1)(c+2)}{2.3} q^c.3pn^2 7_{a-c-3,\,b-1}$. In the fourth case the Probability will be the term $\frac{c(c+1)(c+2)}{2.3} q^c n^3$ multiplied by the lot of the gambler, who wagers to bring forth the event p, by b trials within $a-c-3$ casts, namely $= \frac{c(c+1)(c+2)}{2.3} q^c n^3 7_{a-c-3,\,b}$. The Probability of this fourth case will be therefore

$$= \frac{c(c+1)(c+2)}{2.3} q^c (n^3 7_{a-c-3,\,b} + 3pn^2 7_{a-c-3,\,b-1} + 3np^2 7_{a-c-3,\,b-2} + p^3 7_{a-c-3,\,b-3}).$$

Now the law of progression is clear. The same general coefficients of the terms will be what in the second Problem, the last will be $\frac{c(c+1)\ldots(a-1)}{1.2.3\ldots(a-c)}$. Because it has regard to the terms themselves, the same induction is effective as in Problem 4. The last term will be obtained namely the term $(a-c+1)$, by advancing $n+p$ to the power $a-c$, and there will appear,

$$n^{a-c} 7_{0,b} + (a-c) n^{2-c-1} p 7_{0,b-1} + \frac{(a-c)(a-c-1)}{2} n^{a-c-2} p^2 7_{0,b-2} + p^{a-c} 7_{0,b-a+c}.$$

This clearly follows from the preceding reckonings, by which the general induction is corroborated. Therefore the lot of the gambler generally will be expressed by the following formula,

$$\begin{aligned} 7_{a,b,c} = & q^c (7_{a-c,\,b} + cn 7_{a-c-1,\,b} + p 7_{a-c-1,\,b-1}) \\ & + \frac{c(c+1)}{2} (n^2 7_{a-c-2,b} + 2np 7_{a-c-2,\,b-1} + p^2 7_{a-c-2,\,b-2}) \\ & + \frac{c(c+1)(c+2)}{2.3} (n^3 7_{a-c-3,\,b} + 3n^2 p 7_{a-c-3,\,b-1} \\ & \quad + 3np^2 7_{a-c-3,\,b-2} + p^3 7_{a-c-3,\,b-3}) \ldots \\ & + \frac{c(c+1)(c+2)\ldots(a-1)}{1.2\ldots(a-c)} (n^{a-c} 7_{0,\,b} + (a-c) n^{a-c-1} p 7_{0,\,b-1} \\ & + \frac{(a-c)(a-c-1)}{2} n^{a-c-2} p^2 7_{0,b-2} \ldots + p^{a-c} 7_{0,b-1+c}). \end{aligned}$$

The Celebrated Lagrange offers this same formula. From the general formula of Prob. 2 result all required values of the quantity 7 for use of this formula.

Scholium 1.

16. There should be observed 1) that if $b > a$, there will become $7_{a,b} = 0$, for it is impossible to bring forth the event in b trials within casts fewer in number than b. 2) That if $a = 0$, $b = 0$, it produces $7_{0,0} = 1$, indeed the formula of Problem 2 makes

$p^0 = 1$, as the rule demands, for the victor is the gambler, if no events remain to be brought forth. 3) But if $a = 0$, and b is a positive quantity, it produces $7_{0,b} = 0$, for no casts remain for the gambler. 4) But if a is a positive or null quantity, but b a negative quantity, it produces $7_{a,b} = 1$, for when b is a negative quantity, the event has been exhibited for a number of trials by unity in a greater number than necessary, and hence the gambler to have been the victor. It is impossible a to be a negative quantity, for the last number of the formula is that in which $a = 0$.

Scholium 2.

17. I will illustrate the thing with the following example. A gambler wagers to bring forth the event p, 2 times, the event q, 3 times at least within 6 casts. Here it happens $a = 6$, $b = 2$, $c = 3$, therefore the formula produces

$$\begin{aligned} 7_{6,2,3} =& q^3(7_{3,2} + 3(n7_{2,2} + p7_{2,1}) + 6(n^2 7_{1,2} + 2np7_{1,1} + p^2 7_{1,0}) \\ &+ 10(n^3 7_{0,2} + 3n^2 p7_{0,1} + 3np^2 7_{0,0} + p^3 7_{0,-1}). \end{aligned}$$

But out of the formula Prob. 2 the following values result,

$$\begin{array}{llll} 7_{3,2} = p^2[1 + 2(1-p)], & 7_{2,2} = p^2, & 7_{2,1} = p(1 + 1 - p) & \\ 7_{1,2} = 0, & 7_{1,1} = p, & 7_{1,0} = 1, & \\ 7_{0,2} = 0, & 7_{0,1} = 0, & 7_{0,0} = 1, & 7_{0,-1} = 1 \end{array}$$

By substituting these values, we will have

$$7_{6,2,3} = q^3(60p^2 - 40p^3 - 45p^2 q).$$

The same formula may have been able to be derived by direct calculation. For the cases which present p^2q^3 within 5 casts are $= \frac{1.2.3.4.5}{1.2.3.4.5} = 10$; the cases which present $p^2q^3(1 - p - q) = p^2q^3 - p^3q^3 - p^2q^4$ within 6 casts are

$$\begin{aligned} &1)\ \frac{1.2.3\ldots6}{1.2.3.1.2} - 10 = 50, \\ &2)\ \frac{1.2.3\ldots6}{1.2.3.1.2.3} - 10 = 10, \\ &3)\ \frac{1.2\ldots6}{1\ldots4.1.2} - 10 = 5 \end{aligned}$$

(in any one case I subtract the 10 preceding cases, they may not enter the calculation twice). We will have therefore $60p^2q^3 - 40p^3q^3 - 45p^2q^4$ as above. Indeed this method might be exceedingly lengthy in the cases where a was a large number before b and c.

Scholium 3.

The method employed in this Problem is able to be extended to any number of events. Let, for the sake of an example, there be three events, of which the Probabilities are respectively p, q, r, and the lot of the gambler is sought, who wagers to bring forth the event p, b times at least, the event q, c times at least, the event r, d times at least within a casts, the known lots of the gambler will be put for 2 events, through the general formula of this Problem, and by precisely the same reasoning there will be obtained by supposing $(1-p-q-r)=n$, the formula

$$\begin{aligned}7_{a,b,c,d} =& r^d(7_{a-d,b,c} + d(n7_{a-d-1,b,c} + p7_{a-d-1,b,c} + q7_{a-d-1,b,c-1})\\ &+ \frac{d(d-1)}{2}(n^2 7_{a-d-2,b,c} + 2np7_{a-d-2,b-1,c} + 2nq7_{a-d-2,b,c-1}\\ &\quad + 2pq7_{a-d-2,b-1,c-1} + p^2 7_{a-d-2,b-2,c} + q^2 7_{a-d-2,b,c-2})\\ &+ \frac{d(d+1)(d+2)}{2.3}(n^3 7_{a-d-3,b,c} + 3n^2 p7_{a-d-3,b-1,c}\\ &\quad + 3n^2 q7_{a-d-3,b,c-1} + 3np^2 7_{a-d-3,b-2,c} + 3pq^2 7_{a-d-3,b-1,c-2}\\ &\quad + 6npq7_{a-d-3,b-1,c-1} + 3nq^2 7_{a-d-3,b,c-2} + 3p^2 q7_{a-d-3,b-2,c-1}\\ &\quad + p^3 7_{a-d-3,b-3,c} + q^3 7_{a-d-3,b,c-3}) + \text{etc.}\end{aligned}$$

the coefficient of the last term will be $\frac{d(d+1)\dots(a-1)}{1.2\dots(a-d)}$.

In general let there be any number of events of which the Probabilities are p, q, r, s, $\dots\zeta$, u, and the lot of the gambler is sought who wagers to bring forth the first in b trials, the second in c trials, the third in d trials etc. the last in g trials at least within a casts, there will appear always $7_{a,b,c,d\dots f,g}$ =to a series of terms of which the coefficients will be successively 1, g, $\frac{g(g+1)}{2}$, $\frac{g(g+1)(g+2)}{2.3}$ $\dots$ $\frac{g(g+1)\dots(a-1)}{1.2\dots(a-g)}$. As long as the first term itself will be $7_{a,g,g,c,\text{etc. } f}$: the second will be obtained (by supposing $n = 1-q\dots\text{etc.}-2$) by raising $n+p+q\dots+7$ to the first power, the third will be obtained by raising $(n+p\dots+2)$ to the second power, the fourth will be obtained by raising $(n+p+q\dots+2)$ to the third power and thus successively. But it will be discovered easily by what Probability each and every term should be multiplied, for the sake of an example the term $an^1p^\gamma q^\delta\dots 7^e$ of the term $(m+1)$ must be multiplied by $7_{a-g-b,b-\gamma,c-\delta\,\text{etc. } f-e}$ but the entire series must be multiplied by u^g. All which are demonstrated by repeating, mutatis mutandis, the same reasoning as we employed above. Thus the same reasoning as we employed above. This Problem is solved generally for however many cases.

Problem 9.[16]

19. Let there be placed a vessels successively of which they individually set in line n tickets first white, then black: let be extracted simultaneously from the individual vessels one ticket, thus so that the drawn ticket from the first vessel is restored into the second, from the second into the third, and thus successively, finally from the last into the first, it is sought, what number of tickets first of the whites then of the blacks will be able to be distributed after b drawings.

Solution.

Let us consider first two vessels of which the first contains c black tickets, and the second d. From this second a ticket is extracted; d cases arrive which will present a

[16]*Translator's note*. This is the last problem (Problem VII) of Lagrange. See Section 64.

black ticket, $n-d$ cases which will present a white ticket; that is d cases arrive in which there will be $d-1$ black tickets remaining, $n-d$ cases in which there will be d black tickets remaining in the second vessel. Therefore from this cause the number of black tickets in the second vessel will be probably

$$= \frac{d(d-1)+(n-d)d}{n} = \frac{(n-1)d}{n} = \left(1-\frac{1}{n}\right)d.$$

There will be restored next into this vessel the ticket extracted from the first, but the Probability this ticket to be black is $= \frac{c}{n}$, therefore the number of black tickets will be from this second cause

$$= \left(1-\frac{1}{n}\right)d + \frac{c}{n} = \left(1-\frac{1}{n}\right)d + \frac{n-1}{n},$$
$$\frac{c}{n-1} = \left(1-\frac{1}{n}\right)\left(d+\frac{c}{n-1}\right).$$

From this beginning will be deduced the number of black tickets in the first vessel

$$= \left(1-\frac{1}{n}\right)\left(c+\frac{d}{n-1}\right).$$

Now that the number of black tickets probably contained in a single vessel after b drawings may be evident, I will be calculating as follows. Let there be generally a number p of black tickets in the first vessel, a number q of black tickets in the second vessel thence after a finite number of drawings, after the following extraction the number of black tickets contained in the second vessel will be probably

$$= \left(1-\frac{1}{n}\right)\left(q+\frac{p}{n-1}\right)$$

and the number of black tickets contained in the first vessel will be

$$= \left(1-\frac{1}{n}\right)\left(p+\frac{q}{n-1}\right)$$

by the principle posed above. But the number of black tickets contained in each vessel exists after the first extraction. Therefore by substituting into this formula with the values of the quantities p and q, namely

$$p = \left(1-\frac{1}{n}\right)\left(c+\frac{d}{n-1}\right), \quad q = \left(1-\frac{1}{n}\right)\left(d+\frac{c}{n-1}\right),$$

after reductions the number of black tickets contained in the first vessel will appear after the second extraction,

$$= \left(1-\frac{1}{n}\right)^2\left(c+\frac{2d}{n-1}+\frac{c}{(n-1)^2}\right)$$

and the number of black tickets contained in the second vessel

$$= \left(1 - \frac{1}{n}\right)^2 \left(d + \frac{2c}{n-1} + \frac{d}{(n-1)^2}\right).$$

By substituting again with these values for p and q, the number of black tickets after the third extraction will appear,

in the first vessel

$$\left(1 - \frac{1}{n}\right)\left(c + \frac{3d}{n-1} + \frac{3c}{(n-1)^2} + \frac{d}{(n-1)^3}\right)$$

in the second vessel

$$\left(1 - \frac{1}{n}\right)^3 \left(d + \frac{3c}{n-1} + \frac{3d}{(n-1)^2} + \frac{c}{(n-1)^3}\right).$$

Now the law of progression is clear, just as the number of black tickets is after b drawings

in the first vessel

$$\left(1 - \frac{1}{n}\right)^b \left(c + \frac{b.d}{n-1} + \frac{b.(b-1)}{2(n-1)^2} + \frac{b(b-1)(b-2)}{2.3(n-1)^3}d \ldots + \frac{c \text{ or } d}{(n-1)^b}\right)$$

in the second vessel

$$\left(1 - \frac{1}{n}\right)^b \left(d + \frac{bc}{n-1} + \frac{b(b-1)}{2(n-1)^2}d + \frac{b(b-1)(b-2)}{2.3(n-1)^3}c \ldots + \frac{d \text{ or } e}{(n-1)^b}\right).$$

in the first formula, the numerator of the last term will be d if b is an odd number, and c if b is an even number. In the second formula the numerator of the last term will be c if b is an odd number and d if b is an even number. In other respects the denominators are 1, $n-1$, $(n-1)^2, \ldots (n-1)^b$, the numerators are the binomial coefficients $(1+1)^b$, but the letters $c,\ d$ are alternately assumed.

Now let there be three vessels of which the first contains c black tickets, the second d, the third e. Let generally p be the number of black tickets in any vessel whatsoever, q the number of black tickets in the vessel following after any number of drawings, after the following extraction, the number of black tickets contained in the vessel next in order will be

$$= \left(1 - \frac{1}{n}\right)\left(q + \frac{p}{n-1}\right)$$

as it is evident from the above said. This posed, by assuming $p = d$, $q = e$, the number of black tickets in the third vessel will be after the first extraction

$$= \left(1 - \frac{1}{n}\right)\left(e + \frac{d}{n-1}\right);$$

by supposing $p = c$, $q = d$ the number of black tickets in the second vessel will be after the first extraction

$$= \left(1 - \frac{1}{n}\right)\left(d + \frac{c}{n-1}\right);$$

by supposing $p = e$, $q = c$, the number of black tickets in the first vessel will be after the first extraction

$$= \left(1 - \frac{1}{n}\right)\left(c + \frac{e}{n-1}\right).$$

By substituting again with these values for p and q, after the second extraction the number of black tickets will appear,

in the first vessel

$$\left(1 - \frac{1}{n}\right)^2 \left(c + \frac{2e}{n-1} + \frac{d}{(n-1)^2}\right)$$

in the second vessel

$$\left(1 - \frac{1}{n}\right)^2 \left(d + \frac{2c}{n-1} + \frac{e}{(n-1)^2}\right)$$

in the third vessel[17]

$$\left(1 - \frac{1}{n}\right)^2 \left(e + \frac{2d}{n-1} + \frac{c}{(n-1)^2}\right)$$

In the same manner the number of black tickets appears after the third extraction

in the first vessel

$$\left(1 - \frac{1}{n}\right)^3 \left(c + \frac{3e}{n-1} + \frac{3d}{(n-1)^2} + \frac{c}{(n-1)^3}\right)$$

in the second vessel

$$\left(1 - \frac{1}{n}\right)^3 \left(d + \frac{3c}{n-1} + \frac{3e}{(n-1)^2} + \frac{d}{(n-1)^3}\right)$$

in the third vessel

$$\left(1 - \frac{1}{n}\right)^3 \left(e + \frac{3d}{n-1} + \frac{3c}{(n-1)^2} + \frac{e}{(n-1)^3}\right).$$

Now the law of progression is evident, just as after b drawings the number of black tickets in the third vessel may appear,

$$\left(1 - \frac{1}{n}\right)^b \left(e + \frac{bd}{n-1} + \frac{b(b-1)}{2(n-1)^2}c + \frac{b(b-1)(b-2)}{2.3(n-1)^3}e\right) + \frac{b(b-1)(b-2)(b-3)}{2.3.4(n-1)^4}d \ldots + \frac{c,\ d \text{ or } e}{(n-1)^b}.$$

The coefficients are the same as in the preceding case, and for the terms themselves the original numbers of the black tickets of each and every vase should be written, by starting with a vessel of which the number of tickets is sought and by proceeding in the same order all the way to the end.

In the case of 4, 5, 6 etc. vessels the same coefficients remain, and the terms themselves proceed according to the same law just as in the case of a vessels, if there are $7_{1,0}$, $7_{2,0}$, $7_{3,0}$, $7_{4,0}$ $\ldots 7_{a,0}$ the number of black tickets of the first, second, third $\ldots$etc. a vessel, the number of tickets of the x^i vessel will appear after b drawings

$$\left(1-\frac{1}{n}\right)^b \left(7_{x,0} + \frac{b}{n-1}7_{x-1,0} + \frac{b(b-1)}{2(n-1)^2}7_{x-2,0} + \frac{b(b-1)(b-2)}{2.3(n-1)^3}7_{x-3,0} \text{ etc.}\right)$$

of which series $b+1$ terms should be summed, as the Celebrated Lagrange reports.

Scholium.

20. Of this Problem the particular case is a Problem[18] proposed by the Celebrated Dan. Bernoulli in the *Commentariis Academiae Petropolitanae* for the year 1769. Let there be many vessels of which each contains an equal number of tickets, and in the beginning with the tickets of each and every vessel its color will appear, afterwards the tickets should be led out by the law that the ticket extracted from the one vessel is restored into the next, precisely as in our Problem, this posed, after a given number of drawings, the number of tickets of each and every color probably contained in any one vessel is sought.

Let there be first two vessels of which the first contains n white tickets, the second n black tickets, let r be the number of drawings, there must be in the general formula $b = r$, $d = n$, $c = 0$, and it produces the number of white tickets in the first vessel

$$= n\left(1-\frac{1}{n}\right)^r \left(1 + \frac{r(r-1)}{2(n-1)} + \frac{r(r-1)\ldots(r-3)}{2.3.4(n-1)^4} + \frac{r(r-1)\ldots(r-5)}{2.3\ldots(n-1)^6} \text{ etc.}\right)$$

Let there be $c = n$, $d = 0$, the number of white tickets in the second vessel appears

$$= n\left(1-\frac{1}{n}\right)^r \left(\frac{r}{n-1} + \frac{r(r-1)(r-2)}{2.3(n-1)^3} + \frac{r\ldots(r-4)}{2.3.4.5(n-1)^5} \text{ etc.}\right)$$

Now if n and r quantities should be considered as if large or infinite there will appear

$$\left(1-\frac{1}{n}\right)^r = 1 - \frac{r}{n} + \frac{r^2}{2n^2} - \frac{r^3}{2.3n^3} + \frac{r^4}{2.3.4n^4} \text{ etc. } = e^{-\frac{r}{n}},$$

$$1 + \frac{r(r-1)}{2(n-1)^2} + \frac{r(r-1)(r-2)(r-3)}{2.3.4(n-1)^4} + \frac{r(r-1)\ldots(r-5)}{2.3\ldots6.(n-1)^6} \text{ etc.}$$

$$= 1 + \frac{r^2}{2n^2} + \frac{r^4}{2.3.4n^4} + \frac{r^6}{2.3.4.5.6n^4} + \frac{r^8}{2\ldots8n^8} \text{ etc.}$$

This series has the same form as the preceding, indeed the terms are missing of which the exponents are odd. But since

$$e^{\frac{r}{n}} = 1 + \frac{r}{n} + \frac{r^2}{2n^2} - \frac{r^3}{2.3n^3} + \frac{r^4}{2.3.4n^4} \text{ etc.} = e^{\frac{r}{n}},$$

[18] *Translator's note*. "Disquisitiones analyticae de novo problemate coniecturali," *Novi Commentarii Acad. Petrop.* Vol. XIV, 1769, *pars prior* (1770), pp. 3-25.

it is evident the sum of the sought series to be $\frac{1}{2}\left(e^{\frac{r}{n}}+e^{-\frac{r}{n}}\right)$. Therefore the first formula will become$=\frac{n}{2}\left(1+e^{-\frac{2r}{n}}\right)$ as the Celebrated Bernoulli reports, the second factor of the second is under the same assumption

$$=\frac{r}{n}+\frac{r^3}{2.3n^3}+\frac{r^5}{2\dots 5n^5}\text{ etc.}$$

the sum of this series is as is clear $\frac{1}{2}\left(e^{\frac{r}{n}}-e^{-\frac{r}{n}}\right)$ just as the formula itself is $=\frac{n}{2}\left(1-e^{-\frac{r}{n}}\right)$.

Now let there be three vessels of which the first contains n white tickets, the second n black tickets, the third n red tickets, let r be the number of drawings let there be in the general formula $b=r$, $e=n$, $d=0$, $c=0$, the number of white tickets in the first vessel will be,

$$n\left(1-\frac{1}{n}\right)^r\left(1+\frac{r(r-1)(r-2)}{2.3(n-1)^3}+\frac{r\dots(r-5)}{2\dots(n-1)^6}+\frac{r\dots(r-8)}{2\dots 9(n-1)^9}\text{ etc.}\right)$$

let there be $c=0$, $d=n$, $e=0$, the number of white tickets in the second vessel will be

$$n\left(1-\frac{1}{n}\right)^r\left(r+\frac{r(r-1)\dots(r-3)}{2\dots 4(n-1)^4}+\frac{r\dots(r-6)}{2\dots 7(n-1)^7}+\frac{r\dots(r-9)}{2\dots 10(n-1)^{10}}\text{ etc.}\right)$$

let there be $c=n$, $d=0$, $e=0$, the number of white tickets in the third vessel will be

$$n\left(1-\frac{1}{n}\right)^r\left(\frac{r(r-1)}{2(n-1)^2}+\frac{r\dots(r-4)}{2\dots 5(n-1)^5}+\frac{r\dots(r-7)}{2\dots 8(n-1)^8}\text{ etc.}\right)$$

if now n and r are infinite, the first formula will become

$$ne^{-\frac{r}{n}}\left(1+\frac{r^3}{2.3n^3}+\frac{r^6}{2\dots 6n^6}+\frac{r^9}{2\dots 9n^9}\text{ etc.}\right)$$

let there be

$$1+\frac{r^3}{2.3n^3}+\frac{r^6}{2\dots 6n^6}+\frac{r^9}{2\dots 9n^9}\text{ etc.}=A\left(e^{a\frac{r}{n}}+e^{b\frac{r}{n}}+e^{c\frac{r}{n}}\right),$$

(A, a, b, c, with existence determined,)=

$$\begin{aligned}A\Big(3&+ax+\frac{a^2x^2}{2}+\frac{a^3x^3}{2.3}+\frac{a^4x^4}{2.3.4}\text{ etc.}\Big)\\&+bx+\frac{b^2x^2}{2}+\frac{b^3x^3}{2.3}+\frac{b^4x^4}{2.3.4}\\&+cx+\frac{c^2x^2}{2}+\frac{c^3x^3}{2.3}+\frac{c^4x^4}{2.3.4}\end{aligned}$$

this is for the sake of brevity $x=\frac{r}{n}$ it produces by comparison of terms,

$$\begin{aligned}&A=\frac{1}{3}, && a+b+c=0,\\&a^2+b^2+c^2=0, && a^3+b^3+c^3=3,\\&a^4+b^4+c^4=0, && a^5+b^5+c^5=0,\\&a^6+b^6+c^6=3 && \text{and thus successively.}\end{aligned}$$

Out of the first three equations is obtained $ab+ac+bc=0$, $abc=1$. Therefore the quantities a, b, c will be roots of the equation $a^3-1=0$, namely

$$a=1,\quad b=-\frac{1+\sqrt{-3}}{2},\quad c=-\frac{1-\sqrt{-3}}{2}.$$

Therefore the sum of the series will become

$$=\frac{1}{3}\left(e^{\frac{r}{n}}+e^{-\frac{1-\sqrt{-3}}{2}\frac{r}{n}}+e^{-\frac{1+\sqrt{-3}}{2}\frac{r}{n}}\right).$$

For it will be proved easily from the formula of Newton, to be $a^4+b^4+c^4=0$, $a^5+b^5+c^5=0$, $a^6+b^6+c^6=3$, $a^7+b^7+c^7=0$ and thus successively if it is as in this case

$$ab+ac+bc=0,\quad abc=0,\quad a+b+c=0.$$

This sum is able to be put into the following form

$$\frac{1}{3}e^{\frac{r}{n}}+\frac{2}{3}e^{-\frac{r}{n}}\ \cos\frac{r\sqrt{3}}{2n}.$$

Therefore the formula will become

$$\frac{n}{3}+\frac{2}{3}ne^{-\frac{3r}{2n}}\cos\frac{r\sqrt{3}}{2n},$$

as the Celebrated Bernoulli reports. If n and r are infinite, the second formula becomes

$$ne^{-\frac{r}{n}}\left[\frac{r}{n}+\frac{r^4}{2.3.4n^4}+\frac{r^7}{2\ldots 7n^7}+\frac{r^{10}}{2\ldots 10n^{10}}\text{ etc.}\right].$$

Let there be

$$\begin{aligned}&\frac{r}{n}+\frac{r^4}{2\ldots 4n^4}+\frac{r^7}{2\ldots 7n^7}+\frac{r^{10}}{2\ldots 10n^{10}}\text{ etc.}=Ae^{\frac{ar}{n}}+Be^{\frac{br}{n}}+Ce^{\frac{cr}{n}}\\
=&A\left[1+\frac{ar}{n}+\frac{1}{2}\frac{a^2r^2}{n^2}+\frac{1}{2.3}\frac{a^3r^3}{n^3}+\frac{1}{2.3.4}\frac{a^4r^4}{n^4}\text{ etc.}\right]\\
+&B\left[1+\frac{br}{n}+\frac{1}{2}\frac{b^2r^2}{n^2}+\frac{1}{2.3}\frac{b^3r^3}{n^3}+\frac{1}{2.3.4}\frac{b^4r^4}{n^4}\text{ etc.}\right]\\
+&C\left[1+\frac{cr}{n}+\frac{1}{2}\frac{c^2r^2}{n^2}+\frac{1}{2.3}\frac{c^3r^3}{n^3}+\frac{1}{2.3.4}\frac{c^4r^4}{n^4}\text{ etc.}\right]\end{aligned}$$

it produces by the comparison of terms

$$\begin{aligned}&A+B+C=0, && Aa+Bb+Cc=1,\\
&Aa^2+Bb^2+Cc^2=0, && Aa^3+Bb^3+Cc^3=0,\\
&Aa^4+Bb^4+Cc^4=1, && Aa^5+Bb^5+Cc^5=0\text{ etc.}\end{aligned}$$

Since the series proceeds in the same manner as the preceding, let there be as above

$$a=1,\quad b=-\frac{1+\sqrt{-3}}{2},\quad c=-\frac{1-\sqrt{-3}}{2},$$

they produce the three equations,

$$\begin{aligned}
&A+B+C=0,\\
&A-B\left[\frac{1-\sqrt{-3}}{2}\right]-C\left[\frac{1+\sqrt{-3}}{2}\right]=1,\\
&A^2+B\left[\frac{1-\sqrt{-3}}{2}\right]^2+C\left[\frac{1+\sqrt{-3}}{2}\right]^2=0,
\end{aligned}$$

whence are drawn forth

$$A=\frac{1}{3},\quad B=-\frac{1}{3}\left[\frac{1+\sqrt{-3}}{2}\right],\quad C=-\frac{1}{3}\left[\frac{1-\sqrt{-3}}{2}\right].$$

Therefore the sum will become

$$\frac{1}{3}e^{\frac{r}{n}}-\frac{1}{3}\left[\frac{1+\sqrt{-3}}{2}\right]e^{-\left[\frac{1-\sqrt{-3}}{2}\right]\frac{r}{n}}-\frac{1}{3}\left[\frac{1-\sqrt{-3}}{2}\right]e^{-\left[\frac{1+\sqrt{-3}}{2}\right]\frac{r}{n}}$$

The formula itself will become by using the sine and cosine

$$\frac{n}{3}-\frac{1}{3}ne^{-\frac{3r}{2n}}\cos\frac{r\sqrt{3}}{2n}+\frac{n}{\sqrt{3}}e^{-\frac{3r}{2n}}\sin\frac{r\sqrt{3}}{2n}$$

as the Celebrated Bernoulli reports. If n and r are infinite, the third formula will become

$$ne^{-\frac{r}{n}}\left[\frac{r^2}{2n^2}+\frac{r^5}{2.3.4.5n^5}+\frac{r^8}{2\dots 8n^8}\text{ etc.}\right]$$

Let there be

$$\frac{r^2}{2n^2}+\frac{r^5}{2\dots 5n^5}+\frac{r^8}{2\dots 8n^8}\text{ etc.}=Ae^{\frac{ar}{n}}+Be^{\frac{br}{n}}+Ce^{\frac{cr}{n}}$$

There will be obtained by unfolding,

$$\begin{aligned}
&A+B+C=0, && Aa+Bb+Cc=0,\\
&Aa^2+Bb^2+Cc^2=1, && Aa^3+Bb^3+Cc^3=0,\ \text{etc.}
\end{aligned}$$

since the series proceeds in the same manner as the preceding let there be as above

$$a=1,\quad b=-\frac{1+\sqrt{-3}}{2},\quad c=-\frac{1-\sqrt{-3}}{2},$$

the three prior equations will become

$$\begin{aligned}
&A+B+C=0,\\
&A-B\left[\frac{1-\sqrt{-3}}{2}\right]-C\left[\frac{1+\sqrt{-3}}{2}\right]=0,\\
&A+B\left[\frac{1-\sqrt{-3}}{2}\right]^2+C\left[\frac{1+\sqrt{-3}}{2}\right]^2=1,
\end{aligned}$$

whence are drawn forth

$$A = \frac{1}{3}, \quad B = -\frac{1}{3}\left[\frac{1-\sqrt{-3}}{2}\right], \quad C = -\frac{1}{3}\left[\frac{1+\sqrt{-3}}{2}\right].$$

Therefore the sum will become,

$$\frac{1}{3}e^{\frac{r}{n}} - \frac{1}{3}\left[\frac{1-\sqrt{-3}}{2}\right]e^{-\left[\frac{1-\sqrt{-3}}{2}\right]\frac{r}{n}} - \frac{1}{3}\left[\frac{1+\sqrt{-3}}{2}\right]e^{-\left[\frac{1+\sqrt{-3}}{2}\right]\frac{r}{n}}$$

This formula itself will become by using the sine and cosine,

$$\frac{n}{3} - \frac{n}{3}e^{-\frac{3r}{2n}}\cos\frac{r\sqrt{3}}{2n} - \frac{n}{\sqrt{3}}e^{-\frac{3r}{2n}}\sin\frac{r\sqrt{3}}{2n}$$

as the Celebrated Bernoulli reports. It is easily deduced from the above said in the case of b vessels, the exponents and the coefficients of the quantities of the form $Ae^{c\frac{r}{n}}$ to be derived from the equation $x^b - 1 = 0$ just as the calculation is pressed by no difficulties. And therefore by this I will not be delayed.