DE PROBABILITATE CAUSARUM AB EFFECTIBUS ORIUNDA*

A MATHEMATICAL INQUIRY

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Excellent men of Geometry have treated this material, and especially the Celebrated Laplace in the Memoirs of the Academy at Paris. Since however in Problems to be solved of this kind the higher and difficult analysis will have been applied, I have considered to undertake the price of the trouble to approach the same questions by an elementary method and by proper use of the theory of series. By which reason this other part of the calculation of probabilities is reduced to the theory of combinations, and as I have reduced the first part in the case of the discussion transmitted by the Royal Society.[1] I will attempt to undertake these principal questions here briefly, having been directed chiefly by a method to be illuminated.

§1. *Let there be a vessel containing white and black tickets in infinite number, so that the proportion of white and black tickets is unknown. The extracted tickets are p white q black, the probability of drawing out m white and n black tickets in the future is sought.* The extracted tickets are supposed cast again into the vessel.

Let n' be an integer number of tickets which we shall assume infinite afterwards, it is evident the numbers of white and black tickets to be able to be varied, just as the following table reports:

white tickets	black tickets	white tickets	black tickets
1	$n'-1$	$n'-3$	3
2	$n'-2$	$n'-2$	2
3	$n'-3$	$n'-1$	1
4	$n'-4$		
$\dots$	$\dots$		
$n'-4$	4		

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[1]See Disquisitio Elementaris circa Calculum Probabilium, *Commentationes Societatis Regiae Scientiarum Gottingensis*, Vol. XII, 1793/4, pp. 99-136.

Under this hypothesis the respective Probabilities of p white and q black tickets being drawn proceed,

$$\frac{1^p(n'-1)^q}{n'^{p+q}}, \frac{2^p(n'-2)^q}{n'^{p+q}}, \frac{3^p(n'-3)^q}{n'^{p+q}}, \dots \frac{(n'-3)^p 3^q}{n'^{p+q}},$$
$$\frac{(n'-2)^p 2^q}{n'^{p+q}}, \frac{(n'-1)^p 1^q}{n'^{p+q}},$$

and according to the general rule of the calculation of Probabilities, the corresponding term itself divided by the sum of the terms will be the Probability of that given hypothesis.

This posed, the Probability of extracting m white tickets, and n black will be according to the individual hypotheses respectively,

$$\frac{1^{p+m}(n'-1)^{q+n}}{n'^{p+q+m+n}}, \frac{2^{p+m}(n'-2)^{q+n}}{n'^{p+q+m+n}}, \frac{3^{p+m}(n'-3)^{q+n}}{n'^{p+q+m+n}}, \dots$$
$$\frac{(n'-3)^{p+m}3^{q+n}}{n'^{p+q+m+n}}, \frac{(n'-2)^{p+m}2^{q+n}}{n'^{p+q+m+n}}, \frac{(n'-1)^{p+m}1^{q+n}}{n'^{p+q+m+n}}.$$

Each and every term must be divided by the common divisor, of course the sum mentioned above. I have omitted moreover here, for the sake of brevity the coefficients of the terms, I shall restore these successively.

Therefore the sought Probability will be

$$\frac{\frac{1^{p+m}(n'-1)^{q+n}+2^{p+m}(n'-2)^{q+n}+3^{p+m}(n'-3)^{q+n}\dots+(n'-3)^{p+m}3^{q+n}+(n'-2)^{p+m}2^{q+n}+(n'-1)^{p+m}}{n'^{\,p+q+m+n}}}{\frac{1^p(n'-1)^q+2^p(n'-2)^q+3^p(n'-3)^q\dots+(n'-3)^p3^q+(n'-2)^p2^q+(n'-1)^p}{n'^{\,p+q}}}$$

However let there be for the sake of brevity,

$$1^p + 2^p + 3^p \cdots + (n'-2)^p + (n'-1)^p = \int 1^p, \quad \text{there will be}$$
$$1^p(n'-1)^q + 2^p(n'-2)^q + 3^p(n'-3)^q \cdots$$
$$+ (n'-3)^p 3^q + (n'-2)^p 2^q + (n'-1)^p 1^q =$$
$$n'^q \int 1^p - \frac{q}{1} n'^{q-1} \int 1^{p+1} + \frac{q(q-1)}{1.2} n'^{q-2} \int 1^{p+2} - \frac{q\dots(q-2)}{1\dots3} n'^{q-3} \int 1^{p+3}$$
$$\pm \frac{q\dots(q-2)}{1\dots3} n'^3 \int 1^{p+q-3} \mp \frac{q(q-1)}{1.2} n'^2 \int 1^{p+q-2} \pm \frac{q}{1} n' \int 1^{p+q-1} \mp \int 1^{p+q}.$$

In the same manner it will produce

$$1^{p+m}(n'-1)^{q+n} + 2^{p+m}(n'-2)^{q+n} + 3^{p+m}(n'-3)^{q+n} \cdots$$
$$+ (n'-3)^{p+m}3^{q+n} + (n'-2)^{p+m}2^{q+n} + (n'-1)^{p+m}1^{q+n} =$$
$$n'^{q+n} \int 1^{p+m} - \frac{(q+n)}{1} n'^{q+n-1} \int 1^{p+m+1} + \frac{(q+n)(q+n-1)}{1.2} n'^{q+n-2} \int 1^{p+m+2}$$
$$\pm \frac{(q+n)(q+n-1)}{1.2} n'^2 \int 1^{p+q+m+n-2} \mp \frac{(q+n)}{1} n' \int 1^{p+q+m+n-1} \pm \int 1^{p+q+m+n}$$

But I put n' to infinity, they produce by known formulas (see Jac. Bernoulli *ars conjectandi*) the following equations,

$$\int 1^{p+m} = \frac{n'^{p+m+1}}{p+m+1}, \quad \int 1^{p+m+2} = \frac{n'^{p+m+2}}{p+m+2} \text{ etc.}$$

Therefore the first series being divided by n'^{p+q} produces, $=$

$$n'\left(\frac{1}{p+1} - \frac{q}{1}\frac{1}{p+2} + \frac{q(q-1)}{1.2}\frac{1}{p+3} - \frac{q(q-1)(q-2)}{1.2.3}\frac{1}{p+4}\cdots \right.$$
$$\left. \pm\frac{q(q-1)(q-2)}{1.2.3}\frac{1}{p+q-2} \mp \frac{q(q-1)}{1.2}\frac{1}{p+q-1} \pm \frac{q}{1}\frac{1}{p+q} \mp \frac{1}{p+q+1}\right)$$

Next the other series being divided by $n'^{p+q+m+n}$ produces, $=$

$$n'\left(\frac{1}{p+m+1} - \frac{(q+n)}{1}\frac{1}{p+m+2} + \frac{(q+n)(q+n-1)}{1.2}\frac{1}{p+m+3}\cdots \right.$$
$$\left. \pm\frac{(q+n)(q+n+1)}{1.2}\frac{1}{p+q+m+n-1} \mp \frac{(q+n)}{1}\frac{1}{p+q+m+n} \pm \frac{1}{p+q+m+n+1}\right)$$

Moreover let S be the sum of the first series, I suppose successively $q = 1, 2, 3$ etc. it will produce

$$S = \frac{n'}{(p+1)(p+2)}, \frac{1.2.n'}{(p+1)\cdots(p+3)}, \frac{1.2.3.n'}{(p+1)\cdots(p+4)}$$

whence by analogy it produces in general $S = \frac{1.2\ldots q.n'}{(p+1)\cdots(p+q+1)}$. But the sum of the other series, produces (substituting $p+m$ in place of p, $q+m$ in place of q) $=$

$$\frac{1.2\ldots(q+n)n'}{(p+m+1)\cdots(p+q+m+n+1)}.$$

Now by dividing the other sum by the prior, the sought Probability will be$=$

$$\beta.\frac{1.2.3\ldots(q+n)}{1.2\ldots q}.\frac{(p+1)(p+2)\ldots(p+q+1)}{(p+m+1)\ldots(p+q+m+n+1)} =$$
$$\beta.\frac{(q+1)(q+2)\ldots(q+n)(p+1)(p+2)\ldots(p+q+1)}{(p+m+1)(p+m+2)\ldots(p+q+m+n+1)},$$

β being the coefficient which now we will determine. From the theory of combinations it follows the numerator to be multiplied by the number of combinations $p+q$ of the quantities taken according to the exponent q, or by

$$\frac{(p+q)(p+q-1)\cdots(p+1)}{1.2\ldots q},$$

and by the number of combinations $m+n$ of the quantities taken according to the exponent n, or by

$$\frac{(m+n)(m+n-1)\cdots(m+1)}{1.2\ldots n}.$$

However the denominator must be multiplied in such manner by

$$\frac{(p+q)(p+q-1)\cdots(p+1)}{1.2\ldots q},$$

therefore there will be $\beta =$

$$\frac{(m+n)(m+n-1)\cdots(m+1)}{1.2\ldots n}.$$

Which agrees with the statement of the Celebrated Laplace (*Mem. des Sçavans Étranges* T. 6. §3[2]).

2. The series is able to be summed in another way

$$\frac{1}{p+1} - \frac{q}{1}\frac{1}{p+2} + \frac{q(q-1)}{1.2}\frac{1}{p+3} - \frac{q\cdots(q-2)}{1.2.3}\frac{1}{p+4}\cdots \pm \frac{1}{p+q-1}.$$

There exists

$$1 - \frac{q}{1} + \frac{q(q-1)}{1.2} \text{ etc. } = (1-1)^q = 0.$$

If therefore the series

$$\frac{1}{p+1} - \frac{q}{1}\frac{1}{p+1} + \frac{q(q-1)}{1.2}\frac{1}{p+1} \text{ etc. } = 0$$

is subtracted from the proposed series, the value of it will not be changed, and it will take this form,

$$\frac{q}{p+1}\left(\frac{1}{p+2} - \frac{q-1}{1}\frac{1}{p+3} + \frac{(q-1)(q-2)}{1.2}\frac{1}{p+4}\cdots \pm \frac{1}{p+q+1}\right)$$

subtracting now from the new series

$$\frac{1}{p+2} - \frac{q-1}{1}\frac{1}{p+2} + \frac{(q-1)(q-2)}{1.2}\frac{1}{p+2} \text{ etc. } = 0,$$

our series will take this form,

$$\frac{q(q-1)}{(p+1)(p+2)}\left(\frac{1}{p+3} - \frac{q-2}{p+4} + \frac{(q-2)(q-3)}{1.2}\frac{1}{p+5} \text{ etc.}\right)$$

subtracting again from the new series

$$\frac{1}{p+3} - \frac{q-2}{p+3} + \frac{(q-2)(q-3)}{1.2}\frac{1}{p+3} \text{ etc. } = 0,$$

our series will take this form,

$$\frac{q(q-1)(q-2)}{1.2.3}\left(\frac{1}{p+4} - \frac{q-3}{1}\frac{1}{p+5} + \frac{(q-3)(q-4)}{1.2}\frac{1}{p+6} \text{ etc.}\right)$$

[2] See ”Mémoire sur la Probabilité des causes par les évènements.”

In general if from the series

$$\frac{1}{p+v} - \frac{(q-v+1)}{1}\frac{1}{p+v+1} + \frac{(q-v+1)(q-v)}{1.2}\frac{1}{p+v+2} \cdots \pm \frac{1}{p+q+1}$$

the series

$$\frac{1}{p+v} - \frac{q-v+1}{1}\frac{1}{p+v} + \frac{(q-v+1)(q-v)}{1.2}\frac{1}{p+v} \text{ etc. } = 0$$

is subtracted the first series will take this form,

$$\frac{q-v+1}{p+v}\left(\frac{1}{p+v+1} - \frac{q-v}{1}\frac{1}{p+v+2} + \frac{(q-v)(q-v-1)}{1.2}\frac{1}{p+v+3} \cdots + \frac{1}{p+v+1}\right).$$

Therefore our series will take finally this form

$$\frac{q(q-1)\cdots 2.1}{(p+1)(p+2)\cdots(p+q+1)}$$

as above.

3. If in the vessel of §1 the white tickets were supposed as many as α in total, the following Table will represent the various numbers of white and black tickets:

white tickets	black tickets	white tickets	black tickets
1	$n'-1$	α	$n'-\alpha$
2	$n'-2$		
3	$n'-3$		
4	$n'-4$		

Under this hypothesis the respective Probabilities of extracting p white and q black tickets arise,

$$\frac{1^p(n'-1)^q}{n'^{p+q}}, \frac{2^p(n'-2)^q}{n'^{p+q}}, \frac{3^p(n'-3)^q}{n'^{p+q}} \cdots + \frac{\alpha^p(n'-\alpha)^q}{n'^{p+q}}.$$

Therefore the sum of these terms will exist, using the same method of §1.

$$\begin{aligned}
&n'^q(1^p + 2^p \cdots + \alpha^p) - qn'^{q-1}(1^{p+1} + 2^{p+1} \cdots + \alpha^{p+1})\\
&+ \frac{q(q-1)}{1.2}n'^{q-2}(1^{p+2} + 2^{p+2} \cdots + \alpha^{p+2}) \cdots\\
&\pm \frac{q(q-1)}{1.2}n'^2(1^{p+q-2} + 2^{p+q-2} \cdots + \alpha^{p+q-2})\\
&\mp \frac{q}{1}n'(1^{p+q-1} + 2^{p+q-1} \cdots + \alpha^{p+q-1}) \pm (1^{p+q} + 2^{p+q} \cdots + \alpha^{p+q})
\end{aligned}$$

Now I put α to infinity, it will produce

$$\begin{aligned}
&\frac{n'^q\alpha^{p+1}}{p+1} - qn'^{q-1}\frac{\alpha^{p+2}}{p+2} + \frac{q(q-1)}{1.2}n'^{q-2}\frac{\alpha^{p+3}}{p+3} \cdots\\
&\pm \frac{q(q-1)}{1.2}n'^2\frac{\alpha^{p+q-1}}{p+q-1} \mp \frac{q}{1}n'\frac{\alpha^{p+q}}{p+q} \pm \frac{\alpha^{p+q+1}}{p+q+1}.
\end{aligned}$$

Let $\frac{\alpha}{n'} = \frac{p}{p+q} - \theta$, it produces $\alpha = n'\left(\frac{p}{p+q} - \theta\right)$, omitting therefore the coefficient n'^{p+q+1} which will vanish in the final fraction by aid of the similar coefficient in the denominator, as we have seen §1. the series will produce,

$$\frac{\left(\frac{p}{p+q} - \theta\right)^{p+1}}{p+1} - \frac{q\left(\frac{p}{p+q} - \theta\right)^{p+2}}{p+2} + \frac{\frac{q(q-1)}{1.2}\left(\frac{p}{p+q} - \theta\right)^{p+3}}{p+3}$$

$$\pm \frac{\frac{q(q-1)}{1.2}\left(\frac{p}{p+q} - \theta\right)^{p+q-1}}{p+q-1} \mp \frac{\frac{q}{1}\left(\frac{p}{p+q} - \theta\right)^{p+q}}{p+q} \pm \frac{\left(\frac{p}{p+q} - \theta\right)^{p+q+1}}{p+q+1} =$$

$$\left(\frac{p}{p+q} - \theta\right)^{p+1}\left(\frac{1}{p+1} - \frac{q}{1}\,\frac{\left(\frac{p}{p+q} - \theta\right)}{p+2} + \frac{q(q-1)}{1.2}\,\frac{\left(\frac{p}{p+q} - \theta\right)^2}{p+3} \cdots\right.$$

$$\pm \frac{\frac{q(q-1)}{1.2}\left(\frac{p}{p+q} - \theta\right)^{q-2}}{p+q-1} \mp \frac{\frac{q}{1}\left(\frac{p}{p+q} - \theta\right)^{q-1}}{p+q} \pm \frac{\left(\frac{p}{p+q} - \theta\right)^{q}}{p+q+1}.$$

Let there be made, for the sake of brevity, $\frac{p}{p+q} - \theta = v$, while however let there be $(1-v)^q = 1 - \frac{qv}{1} + \frac{q(q-1)}{1.2}v^2$ etc. let there be made

$$S = \frac{1}{p+1} - \frac{qv}{p+2} + \frac{\frac{q(q-1)}{1.2}v^2}{p+3} - \frac{\frac{q(q-1)(q-2)}{1.2.3}v^3}{p+4} \text{ etc.}$$

from that series let be subtracted

$$\frac{1}{p+1} - \frac{qv}{p+1} + \frac{\frac{q(q-1)}{1.2}v^2}{p+1} - \frac{\frac{q(q-1)(q-2)}{1.2.3}v^3}{p+1} \text{ etc.} = \frac{(1-v)^q}{p+1}$$

it will produce

$$S = \frac{(1-v)^q}{p+1} + \frac{qv}{p+1}S'$$

by supposing

$$S' = \frac{1}{p+2} - \frac{(q-1)v}{p+3} + \frac{(q-1)(q-2)}{1.2}\,\frac{v^2}{p+4} - \frac{(q-1)\ldots(q-3)}{1\ldots3}\,\frac{v^3}{p+5} \text{ etc.}$$

from the same series let

$$\frac{1}{p+2} - \frac{(q-1)v}{p+2} + \frac{(q-1)(q-2)}{1.2}\,\frac{v^2}{p+2} - \frac{(q-1)\ldots(q-3)}{1\ldots3}\,\frac{v^3}{p+2} \text{ etc.} = \frac{(1-v)^{q-1}}{p+2}$$

be subtracted, it will produce

$$S' = \frac{(1-v)^{q-1}}{p+2} + \frac{(q-1)v}{p+2}S'',$$

by supposing

$$S'' = \frac{1}{p+3} - \frac{(q-2)}{1}\,\frac{v}{p+4} + \frac{(q-2)(q-3)}{1.2}\,\frac{v^2}{p+5} \text{ etc.}$$

from this series let

$$\frac{1}{p+3} - \frac{(q-2)}{1}\frac{v}{p+3} + \frac{(q-2)(q-3)}{1.2}\frac{v^2}{p+3} \text{ etc. } = \frac{(1-v)^{q-2}}{p+3}$$

be subtracted, it will produce

$$S'' = \frac{(1-v)^{q-2}}{p+3} + \frac{(q-2)v}{p+3}S'''$$

by supposing

$$S''' = \frac{1}{p+4} - \frac{(q-3)}{1}\frac{v}{p+5} + \frac{(q-3)(q-4)}{1.2}\frac{v^2}{p+6} \text{ etc.}$$

Therefore it will produce

$$\begin{aligned} S &= \frac{(1-v)^q}{p+1} + \frac{qv}{p+1}S' \\ &= \frac{(1-v)^q}{p+1} + \frac{qv(1-v)^{q-1}}{(p+1)(p+2)} + \frac{q(q-1)v^2}{(p+1)(p+2)}S'' \\ &= \frac{(1-v)^q}{p+1} + \frac{qv(1-v)^{q-1}}{(p+1)(p+2)} + \frac{q(q-1)v^2(1-v)^{q-2}}{(p+1)\cdots(p+3)} + \frac{q\ldots(q-2)v^3}{(p+1)\ldots(p+3)}S'''. \end{aligned}$$

Therefore it will be in general as it is known from analogous series,

$$\begin{aligned} S &= \frac{(1-v)^q}{p+1} + \frac{qv(1-v)^{q-1}}{(p+1)(p+2)} + \frac{q(q-1)v^2(1-v)^{q-2}}{(p+1)\ldots(p+3)}\cdots \\ &+ \frac{q(q-1)\ldots 1v^q}{(p+1)(p+2)\ldots(p+q+1)} = \\ &(1-v)^q\left[\frac{1}{p+1} + \frac{q}{(p+1)(p+2)}\left(\frac{v}{1-v}\right) + \frac{q(q-1)}{(p+1)\ldots(p+3)}\left(\frac{v}{1-v}\right)^2\right. \\ &\left.+ \frac{q\ldots(q-2)}{(p+1)\ldots(p+4)}\left(\frac{v}{1-v}\right)^3 \cdots + \frac{q\ldots 1}{(p+1)\ldots(p+q+1)}\left(\frac{v}{1-v}\right)^q\right] = \end{aligned}$$

(by supposing p and q enormous numbers)

$$\begin{aligned} &(1-v)^q\left[\frac{1}{p} + \frac{q}{p^2}\left(\frac{v}{1-v}\right) + \frac{q^2}{p^3}\left(\frac{v}{1-v}\right)^2 + \frac{q^3}{p^4}\left(\frac{v}{1-v}\right)^3 \text{ etc.}\right] = \\ &(1-v)^q\left(\frac{\frac{1}{1-qv}}{1-v}\right) = (1-v)^q\frac{1-v}{(p+q)(1-v)-q} = \frac{(1-v)^{q+1}}{(p+q)(1-v)-q}. \end{aligned}$$

However there is

$$1-v = 1 - \frac{p}{p+q} + \theta = \frac{q}{p+q} + \theta,\ (p+q)(1-v) - q = (p+q)\theta,$$

therefore there exists

$$S = \frac{(1-v)^{q+1}}{(p+q)\theta} = \frac{\left(\frac{q}{p+q}+\theta\right)^{q+1}}{(p+q)\theta}.$$

The sought sum will be therefore

$$\frac{\left(\frac{p}{p+q}-\theta\right)^{p+1}\left(\frac{p}{p+q}+\theta\right)^{q+1}}{(p+q)\theta},$$

namely the first term of the series which the Celebrated Laplace reports (*Mem. sur les Prob. Mem. de Paris* 1778. §18).

4. If it is necessary to approach nearer to the true sum, the following method shall be applied. Let in general the series be

$$\frac{1}{p+1}+\frac{q\alpha}{(p+1)(p+2)}+\frac{q(q-1)\alpha^2}{(p+1)\dots(p+3)}+\frac{q\dots(q-2)\alpha^3}{(p+1)\dots(p+4)}+\frac{q(q-1)\dots\alpha^q}{(p+1)\dots(p+1+1)}$$

the sum of which is sought through approximation, by supposing p and q enormous numbers. The following formulas will appear by division

$$\frac{1}{p+1} = \frac{1}{p}-\frac{1}{p^2}+\frac{1}{p^3}; \qquad \frac{1}{(p+1)(p+2)} = \frac{1}{p^2}-\frac{3}{p^3}+\frac{7}{p^4};$$
$$\frac{1}{(p+1)\dots(p+3)} = \frac{1}{p^3}-\frac{6}{p^4}+\frac{25}{p^5}; \qquad \frac{1}{(p+1)\dots(p+4)} = \frac{1}{p^4}-\frac{10}{p^5}+\frac{65}{p^6};$$
$$\frac{1}{(p+1)\dots(p+8)} = \frac{1}{p^5}-\frac{15}{p^6}+\frac{140}{p^7}; \qquad \frac{1}{(p+1)\dots(p+6)} = \frac{1}{p^6}-\frac{21}{p^7}+\frac{266}{p^8};$$
$$\frac{1}{(p+1)\dots(p+7)} = \frac{1}{p^7}-\frac{28}{p^8}+\frac{322}{p^9}\text{ etc.};$$

and hence our series will take this form,
$\left(\frac{1}{p}-\frac{1}{p^2}+\frac{1}{p^3}\right)+q\alpha\left(\frac{1}{p^2}-\frac{3}{p^3}+\frac{7}{p^4}\right)$
$+(qq-q)\alpha^2\left(\frac{1}{p^3}-\frac{6}{p^4}+\frac{25}{p^5}\right)+(q^3-3q^2-2q)\left(\frac{1}{p^4}-\frac{10}{p^5}+\frac{65}{p^6}\right)\alpha^3$
$+(q^4-6q^3+11q^2)\left(\frac{1}{p^5}-\frac{15}{p^6}+\frac{140}{p^7}\right)\alpha^4$
$+(q^5-10q^4+35q^3)\left(\frac{1}{p^6}-\frac{21}{p^7}+\frac{266}{p^8}\right)\alpha^5$
$+(q^6-15q^5+85q^4)\left(\frac{1}{p^7}-\frac{28}{p^8}+\frac{462}{p^9}\right)\alpha^6$ etc.
$+\frac{1}{p^3}+q\alpha.\frac{7}{p^4}+(qq-q)\alpha^2.\frac{25}{p^5}+(q^3-3q^2+2q)\alpha^3.\frac{65}{p^6}$
$+(q^4-6q^3+11q^2)\alpha^4.\frac{140}{p^7}+(q^5-10q^4+35q^3)\alpha^5.\frac{266}{p^8}$
$+(q^6-15q^5+85q^4)\alpha^6.\frac{462}{p^9}$ etc. However there is
$\frac{1}{p^3}+\frac{7\alpha q}{p^4}+\frac{25\alpha^2q^2}{p^5}+\frac{65\alpha^3q^3}{p^6}+\frac{140\alpha^4q^4}{p^7}+\frac{266\alpha^5q^5}{p^8}+\frac{462\alpha^6q^6}{p^9}$etc.

$= \frac{pp\left(1+\frac{2\alpha q}{p}\right)}{(p-\alpha q)^5} = \frac{pp+2\alpha qp}{(p-\alpha q)^5}$;
$\frac{25\alpha^2 q}{p^5} + \frac{3.65\alpha^3 q^2}{p^6} + \frac{6.140\alpha^4 q^3}{p^7} + \frac{10.266\alpha^5 q^4}{p^8} + \frac{15.462\alpha^6 q^5}{p^9}$ etc. $=$
$\alpha^2 qp^2 \left(\frac{25}{p^7} + \frac{3.65\alpha q}{p^8} + \frac{6.140\alpha^2 q^2}{p^9} + \frac{10.266\alpha^3 q^3}{p^{10}} + \frac{15.462\alpha^4 q^4}{p^{11}}\right.$ etc. $\left.\right)$
$= \frac{25\alpha^2 p^2 q + 20\alpha^3 pq^2}{(p-\alpha q)^7}$;
$\frac{2.65\alpha^3 q}{p^6} + \frac{11.140\alpha^4 q^2}{p^7} + \frac{35.266\alpha^5 q^3}{p^8} + \frac{85.462\alpha^6 q^4}{p^9}$ etc. $=$
$\alpha^3 qp^3 \left(\frac{2.65}{p^9} + \frac{11.140\alpha q}{p^{10}} + \frac{35.266\alpha^2 q^2}{p^{11}} + \frac{85.462\alpha^3 q^3}{p^{12}}\right.$ etc. $\left. =\right)$
$\frac{130\alpha^3 qp^3 + 370\alpha^4 q^2 p^2 + 130\alpha^5 q^3 p}{(p-\alpha q)^9}$;
$\frac{2\alpha^3 q}{p^4} + \frac{11\alpha^4 q^2}{p^5} + \frac{35\alpha^5 q^3}{p^6} + \frac{85\alpha^6 q^4}{p^7}$ etc. $=$
$\alpha^3 qp \left(\frac{2}{p^5} + \frac{11\alpha q}{p^6} + \frac{35\alpha^2 q^2}{p^7} + \frac{85\alpha^3 q^3}{p^8}\right.$ etc.$\left.\right) = \frac{2\alpha^3 qp + \alpha^4 q^2}{(p-\alpha q)^5}$;
$\frac{2.10\alpha^3 q}{p^5} + \frac{11.15\alpha^4 q^2}{p^6} + \frac{35.21\alpha^5 q^3}{p^7} + \frac{85.28\alpha^6 q^4}{p^8}$ etc. $=$
$\alpha^3 qp^2 \left(\frac{2.10}{p^7} + \frac{11.15\alpha q}{p^8} + \frac{35.21\alpha^2 q^2}{p^9} + \frac{85.28\alpha^3 q^3}{p^{10}}\right.$ etc.$\left.\right) =$
$\frac{20\alpha^3 qp^2 + 25\alpha^4 q^2 p}{(p-\alpha q)^7}$. Our series will have therefore the following form,

$$\frac{1}{p-q\alpha}\left(1 - \frac{(p+q\alpha^2)}{(p-q\alpha)^2} + \frac{(pp+2\alpha pq+6pq\alpha^2+2\alpha^3 pq+\alpha^4 q^2)}{(p-q\alpha)^4} - \frac{(25\alpha^2 p^2 q + 20\alpha^3 pq^2 + 20\alpha^3 p^2 q + 25\alpha^4 pq^2)}{(p-q\alpha)^6} + \frac{130\alpha^3 p^3 q + 370\alpha^4 p^2 q^2 + 130\alpha^5 pq^3}{(p-\alpha q)^8} \text{ etc.}\right) \text{ etc.}$$

But in the following manner, I have obtained the sum of these series. In the first series I seek the difference of the coefficients which the following table exhibits:

1		7		25		65		140		266		462		750
	6		18		40		75		126		196		288	
		12		22		35		51		70		92		
			10		13		16		19		22			
				3		3		3		3				

Differences of the fourth order are constants, so that the general term of the series is of known form

$$1 + 6n + \frac{12n(n-1)}{1.2} + \frac{10n(n-1)(n-2)}{1.2.3} + \frac{3n(n-1)(n-2)(n-3)}{1.2.3.4}.$$

This general term contains five kinds of terms multiplied by 1, n, n^2, n^3, n^4. Therefore each and every term will depend on the five preceding terms and by taking A, B, C, D, E

the coefficients to be determined I establish the following equation

$$1+6n+\frac{12n(n-1)}{1.2}+\frac{10n(n-1)(n-2)}{1.2.3}+\frac{3n(n-1)(n-2)(n-3)}{1.2.3.4}=$$
$$A+6A(n-1)+\frac{12A(n-1)(n-2)}{1.2}+\frac{10A(n-1)(n-2)(n-3)}{1.2.3}+\frac{3A(n-1)(n-2)(n-3)(n-4)}{1.2.3.4}$$
$$B+6B(n-2)+\frac{12B(n-2)(n-3)}{1.2}+\frac{10B(n-2)(n-3)(n-4)}{1.2.3}+\frac{3B(n-2)(n-3)(n-4)(n-5)}{1.2.3.4}$$
$$C+6C(n-3)+\frac{12C(n-3)(n-4)}{1.2}+\frac{10C(n-3)(n-4)(n-5)}{1.2.3}+\frac{3C(n-3)(n-4)(n-5)(n-6)}{1.2.3.4}$$
$$D+6D(n-4)+\frac{12D(n-4)(n-5)}{1.2}+\frac{10D(n-4)(n-5)(n-6)}{1.2.3}+\frac{3D(n-4)(n-5)(n-6)(n-7)}{1.2.3.4}$$
$$E+6E(n-5)+\frac{12E(n-5)(n-6)}{1.2}+\frac{10E(n-5)(n-6)(n-7)}{1.2.3}+\frac{3E(n-5)(n-6)(n-7)(n-8)}{1.2.3.4}$$

This equation remains, as it is well known if $n+1$ is written in place of n, in which case the order of each and every term in the series will be greater by unity. If $n+2$ is written in place of n, the order will be greater by two units; and hence any one term in the same manner will depend on the previous five terms, and the scale of the relation will be,

$$1-\frac{A\alpha q}{p}+\frac{B\alpha^2q^2}{p^2}-\frac{C\alpha^3q^3}{p^3}+\frac{D\alpha^4q^4}{p^4}-\frac{E\alpha^5q^5}{p^5}.$$

The same proof succeeds in all cases in whatever kind of series the displayed differences become constants, so that if the differences of the v^{th} order become constants, each and every term depends on the $(v+1)$ preceding terms, whence a recurrent series will appear of which the scale of the relation will contain $v+2$ terms.

In the present case, by aid of the general term I seek the terms 750, 1155, 1715. Now I set up the following five equations,

$$\begin{aligned}170&=1155A+750B+462C+266D+140E\\1155&=750A+462B+266C+140D+65E\\750&=462A+266B+140C+65D+25E\\462&=266A+140B+65C+25D+7E\\266&=140A+65B+25C+7D+E\end{aligned}$$

By eliminating the quantity E, the following four equations appear,

$$\begin{aligned}1400&=714A+315B+110C+24D\\5900&=3038A+1359B+485C+110D\\16135&=8350A+3763B+1359C+315D\\35535&=18445A+8350B+3038C+714D\end{aligned}$$

By eliminating the quantity D, the following three equations appear,

$$\begin{aligned}6200&=2814A+1017B+230C\\17920&=8170A+2971B+678C\\24460&=11186A+4085B+938C\end{aligned}$$

By eliminating the quantity C the following two equations appear

$$41000 = 14396A + 3098B$$
$$94900 = 33376A + 7198B$$

By eliminating the quantity B the following equation appears

$$558900 = 111780A,$$

thence results $A = 5,\ B = -10,\ C = 10,\ D = -5,\ E = 1.$ The scale of the relation will be therefore

$$1 - \frac{5\alpha q}{p} + \frac{10\alpha^2 q^2}{p^2} - \frac{10\alpha^3 q^3}{p^3} + \frac{5\alpha^4 q^4}{p^4} - \frac{\alpha^5 q^5}{p^5} = \left(1 - \frac{\alpha q}{p}\right)^5.$$

But the proposed series is able to take this form,

$$pp(p^{-5} + 7\alpha q p^{-6} + 25\alpha^2 q^2 p^{-7} + 65\alpha^3 q^3 p^{-8} + 140\alpha^4 q^4 p^{-9}$$
$$+ 266\alpha^5 q^5 p^{-10} + 462\alpha^6 q^6 p^{-11} \text{ etc.})$$

However, there exists

$$(p - \alpha q)^{-5} = p^{-5} + 5\alpha q p^{-6} + 15\alpha^2 q^2 p^{-7} + 35\alpha^3 q^3 p^{-8} + 70\alpha^4 q^4 p^{-9} \text{ etc.}$$

Let this series be multiplied through by $a + \frac{6\alpha q}{p}$. and it will produce

$$\begin{array}{llllll} \alpha p^{-5} & +5a\alpha q p^{-6} & +15a\alpha^2 q^2 p^{-7} & +35a\alpha^3 q^3 p^{-8} & +70a\alpha^4 q^4 p^{-9} & \\ & +b & +5b & +15b & +35b & \text{etc.} \end{array}$$

Comparing terms, there is $a = 1,\ 5a + b = 7$, or $b = 2$, and thus it happens sufficient to the other conditions. Therefore the sum will be $\frac{pp+2\alpha qp}{(p-\alpha q)^5}$ as above.

Other series are able to be summed in the same manner. But it should be observed how we have managed more easily to have been able to sum some series, by the figurate number taken of order five, and multiplied by the coefficient to be determined a, with the same numbers multiplied again by the other coefficient to be determined b, and by the same with the previous added, but advanced by one order thus as the first with the second, the second with the third, and thus added in order. While each and every term of the second series is multiplied by the power of the quantity p, by two units higher than in the previous case, I add the figurate number of order seven, and while the second series is able to take this form,

$$\alpha^2 qp^2(25p^{-7} + 3.65\alpha q p^{-8} + 6.140\alpha^2 q^2 p^{-9} + 10.266\alpha^3 q^3 p^{-10} + 15.462\alpha^4 q^4 p^{-11} \text{ etc.}$$

I shall form a new series, namely

$$\begin{array}{llllll} \alpha p^{-7} & +7a\alpha q p^{-8} & +28a\alpha^2 q^2 p^{-9} & +84a\alpha^3 q^3 p^{-10} & +210a\alpha^4 q^4 p^{-11} & \\ & +b & +7b & +28b & +84b & \text{etc.} \end{array}$$

Comparing terms, there exists $a = 25$, $7a + b = 195$, or $b = 20$, and thus it happens sufficient to the other conditions. Therefore the sum will be

$$\frac{25\alpha^2p^2q + 20\alpha^3pq^2}{(p - \alpha q)^7}$$

as above.
The third series is able to take this form

$$\alpha^3qp^3(2.65p^{-9} + 11.140\alpha qp^{-10} + 35.266\alpha^2q^2p^{-11} + 85.462\alpha^3q^3p^{-12} \text{ etc.})$$

But from the figurate numbers taken of order nine, I shall form a new series,

$$\begin{array}{lllll} \alpha p^{-9} & +9a\alpha qp^{-10} & +45a\alpha^2q^2p^{-11} & +165a\alpha^3q^3p^{-12} & +495a\alpha^4q^4p^{-13} \\ & +b & +9b & +45b & +165b \\ & & +c & +9c & +45c \quad \text{etc.} \end{array}$$

Here I have added the third coefficient c, because the first two were not sufficient. Now comparing terms, there exists $a = 130$, $9a+b = 1540$ or $b = 370$, $45a+9b+c = 9310$ or $c = 130$, and thus it happens sufficient to the other conditions. The sum is therefore

$$\frac{130\alpha^3qp^3 + 370\alpha^4q^2p^2 + 130\alpha^5q^3p}{(p - \alpha q)^9}$$

as above.
The fourth series is able to take this form,

$$\alpha^3qp(2p^{-5} + 11\alpha qp^{-6} + 35\alpha^2q^2p^{-7} + 85\alpha^3q^3p^{-8} \text{ etc.})$$

From the figurate numbers taken of order five I shall form a new series

$$\begin{array}{lllll} \alpha p^{-5} & +5a\alpha qp^{-6} & +15a\alpha^2q^2p^{-7} & +35a\alpha^3q^3p^{-9} & +70a\alpha^4q^4p^{-10} \\ & +b & +5b & +15b & +35b \quad \text{etc.} \end{array}$$

By comparing terms, there exists $a = 2$, $5a + b = 11$, or $b = 1$, and thus it happens sufficient to the other conditions. Therefore the sum will be

$$\frac{2\alpha^3qp + 4\alpha^4q^2}{(p - \alpha q)^5}$$

as above.
The fifth series is able to take this form,

$$\alpha^3qp^2(2.10p^{-7} + 11.15\alpha qp^{-8} + 35.21\alpha^2q^2p^{-9} + 85.28\alpha^3q^3p^{-10} \text{ etc.})$$

From the figurate numbers taken of the seventh order, I shall form a new series,

$$\begin{array}{lllll} \alpha p^{-7} & +7a\alpha qp^{-8} & +28a\alpha^2q^2p^{-9} & +84a\alpha^3q^3p^{-10} & +210a\alpha^4q^4p^{-11} \\ & +b & +7b & +28b & +84b \quad \text{etc.} \end{array}$$

By comparing terms, there exists $a = 20$, $7a+b = 165$, or $b = 25$, and thus it happens sufficient to the other conditions. Therefore the sum will be

$$\frac{20\alpha^3 qp^2 + 25\alpha^4 q^2 p}{(p-\alpha q)^7}$$

as above. However the order of the figurate numbers which must be used, will depend on the order of differences which become constants, the first order surpasses the second by unity. Substituting now in place of α its value $\frac{v}{1-v} = \frac{\frac{p}{p+q}-\theta}{\frac{p}{p+q}+\theta}$, it will produce

$$p - \alpha q = \frac{(p+q)\theta}{\frac{p}{p+q}+\theta}; \quad \frac{p + q\alpha^2 = p + q\frac{\left(\frac{p}{p+q}-\theta\right)^2}{\left(\frac{p}{p+q}+\theta\right)^2}}{(p+q)^2\theta^2 =}$$

$$\frac{p\left(\frac{p}{p+q}+\theta\right)^2 + q\left(\frac{p}{p+q}-\theta\right)^2}{(p+q)^2\theta^2} = \frac{pq}{p+q} + \frac{(p+q)\theta^2}{(p+q)^2\theta^2} = \frac{pq+(p+q)^2\theta^2}{(p+q)^3\theta^2}.$$

The formula will become

$$\frac{\left(\frac{p}{p+q}-\theta\right)^{p+1}\left(\frac{p}{p+q}+\theta\right)^{q+1}}{(p+q)\theta}\left(1 - \frac{[pq+(p+q)^2\theta^2]}{(p+q)^3\theta^2}\text{ etc.}\right)$$

(This formula is what the famous Laplace reports §18. of the cited memoir, and §8. Memoir year 1783.)

5. The series of the preceding §

$$\frac{1}{p+1} + \frac{q}{(p+1)(p+2)}\left(\frac{v}{1-v}\right) + \frac{q(q-1)}{(p+1)\dots(p+3)}\left(\frac{v}{1-v}\right)^2 \text{ etc.}$$

converges if $v < 1-v$, that is, if $v < \frac{1}{2}$. Therefore this series serves as long as the selected Probability is demanded from 0 to a fraction inferior to the fraction $\frac{1}{2}$. If the selected Probability is demanded from 0 to a fraction greater to $\frac{1}{2}$ the following method should be used. The Probability is taken from 1 to this fraction greater than $\frac{1}{2}$. To discover this Probability it will be sufficient in the formula of the preceding § to write p in place of q, q in place of p, and $n-\alpha$ in place of α, or to write v in place of $1-v$ and $1-v$ in place of v, whence the series will arise,

$$(1-v)^{q+1}v^p\left[\frac{1}{p+1} + \frac{p}{(q+1)(q+2)}\left(\frac{1-v}{v}\right) + \frac{p(p-1)}{(q+1)\dots(q+3)}\left(\frac{1-v}{v}\right)^2 \text{ etc.}\right]$$
$$= \frac{(1-v)^{q+1}v^{p+1}}{(p+q)v-p},$$

(if p and q are enormous numbers)

$$= \frac{\left(\frac{p}{p+q}+\theta\right)^{p+1}\left(\frac{q}{p+q}-\theta\right)^{q+1}}{-(p+q)\theta},$$

if its value $\frac{p}{p+q} - \theta$ is written for v. This formula will represent the Probability taken from 1 to $\frac{p}{p+q} + \theta$. Therefore the sum of the Probabilities taken from 0 to $\frac{p}{p+q} - \theta$ and from $\frac{p}{p+q} + \theta$ to 1 is

$$\frac{\left(\frac{p}{p+q} - \theta\right)^{p+1} \left(\frac{p}{p+q} + \theta\right)^{q+1} + \left(\frac{p}{p+q} + \theta\right)^{p+1} \left(\frac{p}{p+q} - \theta\right)^{q+1}}{(p+q)\theta}$$

Therefore let K be the Probability taken from 0 to 1, the selected Probability will produce from $\frac{p}{p+q} - \theta$ to $\frac{p}{p+q} + \theta =$

$$K - \frac{\left[\left(\frac{p}{p+q} - \theta\right)^{p+1} \left(\frac{q}{p+q} + \theta\right)^{q+1} + \left(\frac{p}{p+q} + \theta\right)^{p+1} \left(\frac{q}{p+q} - \theta\right)^{q+1}\right]}{(p+q)\theta}$$

as the famous Laplace reports in the Memoir year 1778. §18.

6. If $p = q$, and then $\alpha = 1$, by formula §4. it will happen,

$$\frac{1}{p+1} + \frac{p}{(p+1)(p+2)} + \frac{p(p-1)}{(p+1)\dots(p+3)} + \frac{p\dots(p-2)}{(p+1)\dots(p+4)} \text{ etc.}$$

By supposing $p = 1, 2, 3\dots$ etc. the following Table of p will arise,

$p = 1$	$\frac{1}{2} + \frac{1}{2.3} = \frac{4}{2.3} = \frac{2^2}{2.3}$
$p = 2$	$\frac{1}{3} + \frac{2}{3.4} + \frac{2.1}{3.4.5} = \frac{2.2^4}{3.4.5}$
$p = 3$	$\frac{1}{4} + \frac{3}{4.5} + \frac{3.2}{4.5.6} + \frac{3.2.1}{4.5.6.7} = \frac{384}{4.5.6.7} = \frac{2.3.2^6}{4.5.6.7}$
$p = 4$	$\frac{1}{5} + \frac{4}{5.6} + \frac{4.3}{5.6.7} + \frac{4.3.2}{5.6.7.8} + \frac{4.3.2.1}{5\dots9} = \frac{2.3.4.2^8}{4.5.6.7.8}$
$p = 5$	$\frac{1}{6} + \frac{5}{6.7} + \frac{5.4}{6.7.8} + \frac{5.4.3}{6.7.8.9} + \frac{5\dots2}{6\dots10} + \frac{5\dots1}{6\dots11} = \frac{2.3.4.5.2^{10}}{6\dots11}$

Now it is well known by analogy for $p = p$ to be the sum $= \frac{1.2.3\dots p.2^{2p}}{(p+1)\dots(2p+1)}$. By multiplying this formula by $\frac{1}{2^{2p+1}}$ (because in this case is $v = \frac{1}{2}$) it will be $\frac{1.2\dots p}{2(p+1)\dots(2p+1)}$. But the entire probability is $= \frac{1.2\dots q}{(p+1)\dots(p+q+1)} = \frac{1.2\dots p}{(p+1)\dots(2p+1)}$. Therefore the probability, the number of white tickets not exceeding the number of black tickets is $= \frac{1}{2}$ as it is clear *a priori*.

7. The sum of the Probability taken from 0 to the given quantity β and of the Probability taken from β to 1 must be equal to the Probability taken from 0 to 1. Therefore by the calculation I begin near the preceding § and by preserving the same denominations, the following equation will appear

$$v^{p+1}(1-v)^q \left(\frac{1}{p+1} + \frac{q\alpha}{(p+1)(p+2)} + \frac{q(q-1)\alpha^2}{(p+1)\dots(p+3)} \dots + \frac{q(q-1)\dots 1\alpha^9}{(p+1)\dots(p+q+1)}\right)$$
$$+ v^p(1-v)^{q+1} \left(\frac{1}{q+1} + \frac{\frac{p}{\alpha}}{(q+1)(q+2)} + \frac{\frac{p(p-1)}{\alpha^2}}{(q+1)\dots(q+3)} \dots + \frac{\frac{p(p-1)\dots1}{\alpha^p}}{(q+1)\dots(p+q+1)}\right)$$
$$= \frac{1.2\dots q}{(p+1)\dots(p+q+1)}$$

and hence

$$\frac{1}{p+1}+\frac{q\alpha}{(p+1)(p+2)}+\frac{q(q-1)\alpha^2}{(p+1)\dots(p+3)}\dots+\frac{q(q-1)\dots1\alpha^q}{(p+1)\dots(p+q+1)}$$
$$+\frac{1}{\alpha}\left(\frac{1}{q+1}+\frac{\frac{p}{\alpha}}{(q+1)(q+2)}+\frac{\frac{p(p-1)}{\alpha^2}}{(q+1)\dots(q+3)}\dots+\frac{\frac{p(p-1)\dots1}{\alpha^p}}{(q+1)\dots(p+q+1)}\right)$$
$$=\frac{1.2\dots q}{(p+1)\dots(p+q+1)v^{p+1}(1-v)^q}=\frac{1.2\dots q}{(p+1)\dots(p+q+1)}\frac{(1+\alpha)^{p+q+1}}{\alpha^{p+1}}$$
$$=\frac{1.2\dots p}{(q+1)\dots(p+q+1)}\frac{(1+\alpha)^{p+q+1}}{\alpha^{p+1}}.$$

If $\alpha=1$, the equation will be

$$\frac{1.2\dots p}{(q+1)\dots(p+q+1)}2^{p+q+1}=$$
$$\frac{1}{p+1}+\frac{q}{(p+1)(p+2)}+\frac{q(q-1)}{(p+1)\dots(p+3)}\dots+\frac{q\dots1}{(p+1)\dots(p+q+1)}$$
$$+\frac{1}{q+1}+\frac{p}{(q+1)(q+2)}+\frac{p(p-1)}{(q+1)\dots(q+3)}\dots+\frac{p\dots1}{(q+1)\dots(p+q+1)}$$

or because of

$$\frac{1.2\dots q}{(p+1)\dots(p+q+1)}=\frac{1.2\dots p}{(p+1)\dots(p+q+1)}$$
$$\frac{1.2\dots p}{(p+1)\dots(p+q+1)}2^{p+q+1}=$$

$$\frac{1.2\dots p}{(q+1)\dots(p+q+1)}\left[\begin{array}{l}1+\frac{(p+q+1)}{1}+\frac{(p+q+1)(p+q)}{1.2}\dots+\frac{(p+q+1)\dots(p+2)}{1.2\dots q}\\+1+\frac{(p+q+1)}{1}+\frac{(p+q+1)(p+q)}{1.2}\dots+\frac{(p+q+1)\dots(q+2)}{1.2\dots q}\end{array}\right]$$

$$\frac{1.2\dots p}{(q+1)\dots(p+q+1)}\left[\begin{array}{llll}1+\frac{(p+q+1)}{1} & +\frac{(p+q+1)(p+q)}{1.2} & \dots+\frac{(p+q+1)\dots(p+2)}{1.2\dots q}\\+1+\frac{(p+q+1)}{1} & +\frac{(p+q+1)(p+q)}{1.2} & \dots+\frac{(p+q+1)\dots(p+2)}{1.2\dots q}\\+\frac{(p+q+1)\dots(p+1)}{1.2\dots(q+1)} & +\frac{(p+q+1)\dots p}{1.2\dots(q+2)}\dots & +\frac{(p+q+1)\dots(q+2)}{1.2\dots p}\end{array}\right]=$$

$$\frac{1.2\dots p}{(q+1)\dots(p+q+1)}2\begin{array}{l}\left[1+\frac{(p+q+1)}{1}+\frac{(p+q+1)(p+q)}{1.2}\dots+\frac{(p+q+1)\dots(p+2)}{1.2\dots q}\right]\\+\frac{1}{q+1}+\frac{p}{(q+1)(q+2)}+\frac{p(p-1)}{(q+1)\dots(q+3)}\dots+\frac{p(p-1)\dots[p-(p-q-2)]}{(q+1)\dots(q+p-q)}\end{array}.$$

Therefore it is

$$\frac{1}{p+1}+\frac{q}{(p+1)(p+2)}+\frac{q(q-1)}{(p+1)\dots(p+3)}\dots+\frac{q(q-1)\dots1}{(p+1)\dots(q+p+1)}=$$
$$\frac{1.2\dots p}{(q+1)\dots(p+q+1)}2^{p+q}-\frac{1}{2}\left(\frac{1}{q+1}+\frac{p}{(q+1)(q+2)}+\frac{p(p-1)}{(q+1)\dots(q+3)}\dots\right.$$
$$\left.+\frac{p(p-1)\dots p-(p-q-2)}{(q+1)\dots(q+p-q)}\right).$$

By supposing $\alpha = 1$, let $v = \frac{1}{2}$, $1 - v = \frac{1}{2}$, now our formula will be multiplied by $v^{p+1}(1-v)^q$ or by $\frac{1}{2^{p+q+1}}$. Dividing next by $\frac{1.2\ldots p}{(q+1)\ldots(p+q+1)}$ it will show the Probability the number of white tickets will not be exceeding the number of black tickets, and it will be =

$$\frac{1}{2} - \frac{1}{2^{p+q+1}}\frac{(q+1)\ldots(p+q+1)}{1\ldots p} \times \left(\frac{1}{q+1} + \frac{p}{(q+1)(q+2)} + \frac{p(p-1)}{(q+1)\ldots(q+3)}\cdots + \frac{p(p-1)\ldots[p-(p-q-2)]}{(p+1)\ldots(q+p-q)}\right).$$

8. This last formula is able to serve in the cases where $p - q$ is a small number, in which case the first formula frequently diverges. Of this kind the Celebrated Laplace handles an example in the Memoir year 1783 §18. There exists in Burgundy a little city named Viteaux, in which are born during one year 415 infants, 203 of male 212 of female kind. Let $p = 212,\ q = 203$ be made, they produce $p+q = 415$, $p+q+1 = 416$. The series

$$1 + \frac{p}{q+2} + \frac{p(p-1)}{(q+2)(q+3)}\cdots + \frac{p(p-1)\ldots[p-(p-q-2)]}{(q+1)\ldots(q+p-q)} =$$
$$1 + \frac{212}{205} + \frac{212.211}{205.206} + \frac{212.211.210}{205.206.207} + \frac{212.211.210.209}{205.206.207.208} + \frac{212.211.210}{205.206.207} + \frac{212.211}{205.206} + \frac{212}{205} + 1 = 9.4630.$$

Now there will be from the theorem of Stirling

$$\frac{(q+1)\ldots(p+q+1)}{1.2\ldots p} = \frac{(p+q+1)^{p+q+\frac{3}{2}}}{\sqrt{2\pi}p^{p+\frac{1}{2}}q^{q+\frac{1}{2}}e}$$

(e is the number of which the hyperbolic logarithm is unity). Thus the calculation proceeds,

$416\frac{1}{2}\log 416$	=	1090.8523594	
$\log 9.4630$	=	8.6663986	
204		1089.5187580	1089.5187580
$212\frac{1}{2}\log 212$	=	494.3463787	1090.2847088
$203\frac{1}{2}\log 203$	=	469.5754360	9.2340492
$417\,l2$	=	125.5295100	0.17141
$l\sqrt{2\pi}$	=	0.3990899	
$l\,e$	=	0.4342942	
		1090.2847088	

Therefore the Probability of the number of girls not being in excess of the number of boys will be $= 0.5 - 0.17141 = 0.32859$, and the Probability of the number of girls being in excess of the number of boys will be $= 0.67141$, just as the Celebrated Laplace reports.

9. By the quantity α remaining indeterminate, the general equation is able to take the following form

$$\frac{1.2\dots q}{(p+1)\dots(p+q+1)}\frac{(1+\alpha)^{p+q+1}}{\alpha^{p+1}} =$$
$$\frac{1.2\dots q}{(p+1)\dots(p+q+1)}\left(\alpha^q + \frac{p+q+1)}{1}\alpha^{q-1} + \frac{(p+q+1)(p+q)}{1.2}\alpha^{q-2}\dots\right.$$
$$+\left.\frac{(p+q+1)\dots(p+2)}{1.2\dots q}\right) + \frac{1}{\alpha^{p+1}} + \frac{(p+q+1)}{1}\cdot\frac{1}{\alpha^p}$$
$$+\frac{(p+q+1)(p+q)}{1.2}\cdot\frac{1}{\alpha^{p-1}}\dots + \frac{(p+q+1)\dots(p+2)}{1.2\dots q}\frac{1}{\alpha^{p-q+1}}$$
$$+\frac{(p+q+1)\dots(p+1)}{1.2\dots(q+1)}\frac{1}{\alpha^{p-q}} + \frac{(p+q+1)\dots p}{1.2\dots(q+2)}\frac{1}{\alpha^{p-q-1}}$$
$$\left.\frac{(p+q+1)\dots(p+2)}{1.2\dots q}\frac{1}{\alpha}\right).$$

The method of summation of these various series by approximation requires contemplations, which here it was set forth at greater length, and which deserve to be set forth apart.

10. *Let there be some vessel containing an infinite number of white and black tickets, there are extracted from that vessel* p *white* q *black tickets, the Probability is sought* m *white and* $2a-m$ *black tickets will be exiting in a following operation*. From §1. the Probability of m white, n black tickets exiting from the vessel whence already exited there are p white tickets, q black, is =

$$\frac{(m+1)(m+2)\dots(m+n)}{1.2\dots n}\frac{(q+1)(q+2)\dots(q+n)(p+1)(p+2)\dots(p+q+1)}{(p+m+1)(p+m+2)\dots(p+q+m+n+1)}$$

by substituting the quantity $2a-m$ in place of n, this formula is =

$$\frac{(m+1)(m+2)\dots 2a}{1.2\dots(2a-m)}\frac{(q+1)(q+2)\dots(q+2a-m)(p+1)(p+2)\dots(p+q+1)}{(p+m+1)(p+m+2)\dots(p+q+2a+1)}$$
$$=\frac{1.2.3\dots 2a}{1.2\dots m\,1.2\dots(2a-m)}\frac{1.2\dots(q+2a-m)}{1.2\dots q}\frac{1.2\dots(p+q+1)}{1.2\dots p}\frac{1.2\dots(p+m)}{1.2\dots(p+q+2a+1)}$$
$$=\frac{1.2\dots a}{1.2\dots(p+q+2a+1)}\frac{1.2\dots(p+q+1)}{1.2\dots p.1.2\dots q}\frac{1.2\dots(q+2a-m)}{1.2\dots(2a-m)}\frac{1.2\dots p+m}{1.2\dots m}$$

Now in order to obtain the Probability of the number of white tickets to not be in excess of the number of black tickets, the sum of all these values should be taken from $m=1$ to $m=a$. However constant is the coefficient

$$\frac{1.2.3\dots 2a}{1.2\dots(p+q+2a+1)}\cdot\frac{1.2.3\dots(p+q+1)}{1.2\dots p.1.2\dots q},$$

therefore it suffices to consider the formula

$$\frac{1.2\dots(q+2a-m)}{1.2\dots(2a-m)}\cdot\frac{1.2\dots p+m}{1.2\dots m} =$$
$$(2a-m+1)(2a-m+2)\dots(q+2a-m)(m+1)(m+2)\dots(p+m) = 7_m.$$

Let the following term be $= 7_{m+1}$there will exist the equation

$$7_m = \frac{(1+2a-m)(m+1)}{(2a-m)(p+m+1)} 7_{m+1}.$$

However there is $7_{m+1} = 7_m + \Delta 7_m$, ($\Delta$ is the sign of a finite difference). Hence it produces,

$$7_m \left(1 - \frac{(m+1)(q+2a-m)}{(2a-m)(p+m+1)}\right) = \frac{(m+1)(q+2a-m)}{(2a-m)(p+m+1)} \Delta 7_m,$$

or

$$7_m = \frac{(m+1)(q+2a-m)}{2ap-q-m(p+q)} \Delta 7_m.$$

In a similar way the equations will exist

$$7_{m-1} = \frac{m(q+2a+1-m)}{2ap+p-m(p+q)} \Delta 7_{m-1},$$

$$7_{m-2} = \frac{(m-1)(q+2a+2-m)}{2ap+p-(m+2)(p+q)} \Delta 7_{m-2}, \text{ etc.}$$

If it happens $7_{m-1} = \alpha \Delta 7_{m-1}$, $7_{m-2} = \alpha \Delta 7_{m-2}$ etc. (by α proving to be constant) it will produce

$$7_{m-1} + 7_{m-2} + 7_{m-3} \text{ etc. } = \alpha(\Delta 7_{m-1} + \Delta 7_{m-2} + \Delta 7_{m-3} + \text{ etc.}) = \alpha 7_m$$

or

$$\Sigma 7_{m-1} = \frac{m(q+2a+1-m)}{2ap+p-m(p+q)} \, 7_m$$

(Σ is the sign of the summation of finite differences) but by supposing $m = a$, it makes

$$\Sigma 7_{a-1} = \frac{a(q+a+1)}{p+a(p-q)} \, 7a.$$

But it makes, if the formula of the approximation of Stirling is applied, and it is supposed $m = n = a$, it makes I say,

$$\frac{\gamma(q+a)^{q+a+\frac{1}{2}}(p+a)^{p+a+\frac{1}{2}}(q+q)^{p+q+\frac{3}{2}}}{p^{p+\frac{1}{2}}q^{q+\frac{1}{2}}(p+q+2a)^{p+q+2a+\frac{3}{2}}} =$$

$$\frac{\gamma\sqrt{(p+a)(q+a)}(p+q)^{\frac{3}{2}}(q+a)^{q+a}(p+a)^{p+a}(p+q)^{p+q}}{\sqrt{pq}(p+q+2a)^{\frac{3}{2}}(p+q+2a)^{p+q+2a}p^p q^q} =$$

$$\frac{\gamma\sqrt{(p+a)(q+a)}(p+q)^{\frac{3}{2}}(\frac{q+a}{p+q+2a})^{q+a}(\frac{p+a}{p+q+2a})^{p+a}(p+q)^{p+q+2a}}{\sqrt{pq}(p+q+2a)^{\frac{3}{2}}(p+q)^{2a}p^p q^q} =$$

$$\frac{\gamma p^a q^a (p+q)^{\frac{3}{2}}\sqrt{(p+a)(q+a)}}{(p+q)^{2a}\sqrt{pq}(p+q+2a)^{\frac{3}{2}}} \left(\frac{(p+a)(q+a)}{q(p+q+2a)}\right)^{q+a} \left(\frac{(p+q)(p+a)}{p(p+q+2a)}\right)^{p+a} =$$

$$\frac{\gamma p^a q^a (p+q)^{\frac{3}{2}}\sqrt{(p+a)(q+a)}}{(p+q)^{2a}\sqrt{pq}(p+q+2a)^{\frac{3}{2}}} \left(1 + \frac{a(p-q)}{q(p+q+2a)}\right)^{q+a} \left(1 - \frac{a(p-q)}{p(p+q+2a)}\right)^{p+a}$$

as the Celebrated Laplace refers in the Memoir year 1778. §20. Here for the sake of brevity I have supposed $\gamma = \frac{1.2...2a}{(1.2.3...a)^2}$.

Let the equation

$$7_m = \frac{(m+1)(q+2a-m)}{2ap-q-m(p+q)}\Delta 7_m,$$

be considered again and by successively attributing to the quantity m the values $0,\ 1,\ 2\ldots a$, the following equations exist:

$$7_0 = \frac{(q+2a)}{2ap-q}\Delta 7_0, \quad 7_1 = \frac{2(q+2a-1)}{(2a-1)p-2q}\Delta 7_1, \quad 7_2 = \frac{3(q+2a-2)}{(2a-2)p-3q}\Delta 7_2,$$
$$7_3 = \frac{4(q+2a-3)}{(2a-3)p-4q}\Delta 7_3 \ldots 7_a = \frac{a(q+a+1)}{(a+1)p-aq}\Delta 7_a.$$

Certainly

$$7_1 = 7_0 + \Delta 7_0 = \left(\frac{q+2a}{2ap-q}+1\right)\Delta 7_0 = \frac{2a(p+1)}{2ap-q}\Delta 7_0,$$
$$7_2 = 7_1 + \Delta 7_1 = \left(\frac{2(q+2a-1)}{(2a-1)p-2q}+1\right)\Delta 7_1 = \frac{(2a-1)(p+2)}{(2a-1)p-2q}\Delta 7_1;$$
$$7_3 = 7_2 + \Delta 7_2 = \left(\frac{3(q+2a-2)}{(2a-2)p-3q}+1\right)\Delta 7_2 = \frac{(2a-2)(p+3)}{(2a-2)p-3q}\Delta 7_2$$

or,

$$7_0 = \frac{(q+2a)}{2ap-q}\Delta 7_0, \quad 7_1 = \frac{2a(p+1)}{2ap-q}\Delta 7_0, \quad 7_2 = \frac{(2a-1)(p+2)}{(2a-1)(p-2q}\Delta 7_1,$$
$$7_3 = \frac{(2a-2)(p+3)}{(2a-2)(p-3q}\Delta 7_2, \quad 7_4 = \frac{(2a-3)(p+4)}{(2a-3)p-4q}\Delta 7_3 \ldots$$
$$7_a = \frac{(a+1)(p+a)}{(a+1)p-aq}\Delta 7_{a-1}.$$

But there is

$$\Delta 7_1 = 7_2 - 7_1 = \frac{(2a-1)(p+2)}{(2a-1)p-2q}\Delta 7_1 - \frac{2a(p+1)}{2ap-q}\Delta 7_0.$$

Therefore

$$\frac{2a(p+1)}{2ap-q}\Delta 7_0 = \frac{2(2a+q-1)}{(2a-1)p-2q}\Delta 7_1.$$

Likewise

$$\frac{\Delta 7_1}{(2a-1)p-2q} = \frac{2a(p+1)}{1.2(2ap-q)(2a+q-1)}\Delta 7_0.$$

Therefore

$$7_2 = \frac{2a(2a-1)(p+2)(p+1)}{1.2(2ap-q)(2a+q-1)}\Delta 7_0.$$

In addition

$$\Delta 7_2 = 7_3 - 7_2 = \frac{(2a-2)(p+3)}{(2a-2)p-3q}\Delta 7_2 - \frac{2a(2a-1)(p+2)(p+1)}{1.2(2ap-q)(2a+q-1)}\Delta 7_0.$$

Therefore

$$\frac{2a(2a-1)(p+2)(p+1)}{1.2(2ap-q)(2a+q-1)}\Delta 7_0 = \frac{3(2a+q-2)}{(2a-2)p-3q}\Delta 7_2,$$
$$\frac{\Delta 7_2}{(2a-2)p-3q} = \frac{2a(a-1)(p+2)(p+1)}{1.2.3(2ap-q)(2a+q-1)(2a+q-2)}\Delta 7_0,$$
$$7_3 = \frac{2a(2a-1)(2a-2)(p+3)(p+2)(p+1)}{1.2.3(2ap-q)(2a+q-1)(2a+q-2)}\Delta 7_0,$$

and hence

$$7_0 = \frac{q+2a}{2ap-q}\Delta 7_0, \quad 7_1 = \frac{2a(p+1)}{2ap-q}\Delta 7_0, \quad 7_2 = \frac{2a(2a-1)(p+2)(p+1)}{1.2(2ap-q)(2a+q-1)}\Delta 7_0,$$
$$7_3 = \frac{2a(2a-1)(2a-2)(p+3)(p+2)(p+1)}{1.2.3(2ap-q)(2a+q-1)(2a+q-2)}\Delta 7_0$$
$$7_4 = \frac{2a(2a-1)(2a-2)(2a-3)(p+4)(p+3)(p+2)(p+1)}{1.2.3.4(2ap-q)(2a+q-1)(2a+q-2)(2a+q-3)}\Delta 7_0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$7_a = \frac{2a(2a-1)\ldots(a+1)(p+a)(p+a-1)\ldots(p+1)}{1.2.3\ldots a(2ap-q)(2a+q-1)(2a+q-2)\ldots(a+q+1)}\Delta 7_0.$$

Therefore

$$7_0 + 7_1 + 7_2 + 7_3 + 7_4 \cdots + 7_a = \frac{q+2a}{2ap-q}\Delta 7_0$$
$$+ \frac{\Delta 7_0}{2ap-q}\left(\frac{2a(p+1)}{1} + \frac{2a(2a-1)(p+2)(p+1)}{1.2\ldots(2a+q-1)}\right.$$
$$+ \frac{2a(2a-1)(2a-2)(p+3)(p+2)(p+1)}{1.2.3(2a+q-1)(2a+q-2)}$$
$$+ \frac{2a(2a-1)(2a-2)(2a-3)(p+4)(p+3)(p+2)(p+1)}{1.2.3.4(2a+q-1)(2a+q-2)(2a+q-3)}\cdots$$
$$\left.+ \frac{2a(2a-1)\ldots(a+1)(p+a)(p+a-1)\ldots(p+1)}{1.2.3\ldots a(2ap+q-1)(2a+q-2)\ldots(a+q+1)}\right)$$

This inverted series will take the following form,

$$7_a\left(1 + \frac{a(a+q-1)}{(p+a)(a+1)} + \frac{a(a-1)(a+q+1)(a+q+2)}{(p+a)(p+a-1)(a+1)(a+2)}\right.$$
$$\left.+ \frac{a(a-1)(a-2)(a+q+1)(a+q+2)(a+q+3)}{(p+a)(p+a-1)(p+a-2)(a+1)(a+2)(a+3)}\text{ etc.}\right) =$$

(if p and q are supposed enormous numbers)

$$7_a\left(1+\frac{a(a+q-1)}{(p+a)(a+1)}+\frac{a^2(a+q+1)^2}{(p+a)^2(a+1)^2}+\frac{a^3(a+q+1)^3}{(p+a)^3(a+1)^3}\text{ etc.}\right)=$$
$$7_a\left(\frac{1}{1-\frac{a(a+q+1)}{(p+a)(a+1)}}\right)=7_a\frac{(p+a)(a+1)}{(p+a)(a+1)-a(a+q-1)}=7_a\frac{(p+a)(a+1)}{p+a(p-q)}.$$

11. If I wish to progress further, I would observe the series anew,

$$1+\frac{a(a+q-1)}{(p+a)(a+1)}+\frac{a(a-1)(a+q+1)(a+q+2)}{(p+a)(p+a-1)(a+1)(a+2)}\text{ etc.}$$

and there will proceed

$$\frac{(a-1)(a+q+2)}{(p+a-1)(a+2)}=\frac{\frac{a(a+q+1)}{(p+a)(a+1)}-\frac{q+2}{(p+a)(a+1)}}{1+\frac{p-2}{(p+a)(a+1)}}=$$
$$\frac{a(a+q+1)}{(p+a)(a+1)}-\left(\frac{(p-2)a(a+q+1)}{(a+a)^2(a+1)^2}+\frac{q+2}{(p+a)(a+1)}\right)=$$
$$\frac{a(a+q+1)}{(p+a)(a+1)}-\frac{(aap+2apq+aaq)}{(p+a)^2(a+1)^2}$$

(by neglecting quantities in which the sum of the set forth quantities a, p, q is smaller than unity). In the same manner

$$\frac{(a-2)(a+q+3)}{(p+a-2)(a+3)}=\frac{a(a+q+1)}{(p+a)(a+1)}-\frac{2(aap+2pq+aaq)}{(p+a)^2(a+1)^2};$$
$$\frac{(a-3)(a+q+4)}{(p+a-3)(a+4)}=\frac{a(a+q+1)}{(p+a)(a+1)}-\frac{3(aap+2pq+aaq)}{(p+a)^3(a+1)^3},$$

and so forth. Let, for the sake of brevity,

$$A=\frac{a(a+q+1)}{(p+a)(a+1)},\quad B=\frac{aap+2apq+aaq}{(p+a)^2(a+1)^2},$$

and by neglecting squares of the quantity B, the sought series will be $=$

$$1+A+A(A-B)+A(A-B)(A-2B)+A(A-B)(A-2B)(A-3B)\text{ etc.'}$$
$$=1+A+AA+A^3+A^4+A^5\text{ etc. }-AB(1+3A+6A^2+10A^2\text{ etc.})$$
$$=\frac{1}{1-A}-\frac{AB}{(1-A)^3}=\frac{1}{1-A}\left(1-\frac{AB}{(1-A)^2}\text{ etc.}\right)=$$
$$\frac{(a+p)(a+1)}{ap+p-aq}\left(1-\frac{a^2(a+q+1)(ap+2pq+aq)}{(ap+p-aq)(p+a)^2(a+1)^2}\text{ etc.}\right).$$

The Celebrated Laplace reports a similar series §20, and the higher terms of this series can be discovered, the laborious calculation will become only prolix.

12. The Celebrated Laplace in the *Mémoires Etrangers* T. 6 attacks the following Problem. *Two gamesters A and B of whom skill is unknown, between themselves on the same condition that he who first will have won v games, obtains the sum 1. However the gamesters are forced to cease the game, while gamester A lacks f games, gamester B h games. With this put it is sought in what way the sum is divided between the gamesters.*

If the skill of the gamester A was p, of the gamester B $1-p$, and the quantity p had been known, the Problem was able to be reported to the second Problem among the most general Problems which the Celebrated Lagrange has handled, (*Mem. Acad. Berlin* 1775) and which I have demonstrated by an elementary method in the memoir transmitted in the Royal Society. The formula reported at the end of this Problem is this

$$p^a\left(1+\frac{a(1-p)}{p}+\frac{a(a-1)}{1.2}\frac{(1-p)^2}{p^2}\cdots+\frac{a(a-1)\dots(b+1)}{1\dots(a-b)}\frac{1-p^{a-b}}{p^{a-b}}\right).$$

Here there must be $a=f+h-1$, $b=h$. But I prove in §1, that if there are p white tickets q black drawn from a vessel containing an infinite number of white and black tickets, the Probability of extracting in the future m white tickets, n black, will be

$$\frac{(q+1)(q+2)\dots(q+n)(p+1)(p+2)\dots(p+q+1)}{(p+m+1)(p+m+2)\dots(p+q+m+n+1)}.$$

Here there must be $p=v-f$, $q=v-h$, thus the formula becomes

$$\frac{(v-h+1)(v-h+2)\dots(v-h+n)(v-f+1)(v-f+2)\dots(2v-f-h+1)}{(v-f+m+1)(v-f+m+2)\dots(2v-f-h+m+n+1)}.$$

Now into the preceding series, in place of the first term p^a this formula is substituted by supposing $m=0$, $n=f+h-1$,whence the term will produce

$$\frac{(v-h+1)(v-h+2)\dots(v+f-1)(v-f+1)(v-f+2)\dots(2v-f-h+1)}{(v-f+1)(v-f+2)\dots(2v)}=$$

$$\frac{(v-h+1)(v-h+2)\dots(v+f-1)}{(2v-f-f+2)(2v-f-h+3)\dots(2v)}.$$

In the place of the second term $p^{a-1}(1-p)$ this same formula is substituted by supposing $m=1$, $n=f+h-2$, whence the term will produce

$$\frac{(v-h+1)(v-h+2)\dots(v+f-2)(v-f+1)(v+f+2)\dots(2v-f-h+1)}{(v-f+2)(v-f+3)\dots(2v)}$$

which is the term of the first case multiplied by $\frac{v-f+1}{v+f-1}$. In the place of the third term $p^{a-2}(1-p)^2$ this same formula is substituted by supposing $m=2$, $n=f+h-3$, whence the term will produce

$$\frac{(v-h+1)(v-h+2)\dots(v+f-3)(v-f+1)(v+f+2)\dots(2v-f-h+1)}{(v-f+3)(v-f+4)\dots(2v)}$$

which is the term of the first case multiplied by $\frac{(v-f+1)(v-f+2)}{(v+f-1)(v-f-2)}$. In the place of the fourth term $p^{a-3}(1-p)^3$ this same formula is substituted by supposing $m=3$, $n=f+h-4$, whence the term will produce

$$\frac{(v-h+1)(v-h+2)\dots(v+f-4)(v-f+1)(v+f+2)\dots(2v-f-h+1)}{(v-f+4)(v-f+5)\dots(2v)}$$

which is the term of the first case multiplied by $\frac{(v-f+1)(v-f+2)(v-f+3)}{(v+f-1)(v-f-2)(v+f-3)}$.
In general in place of the term $(a-b+1)^1$, which is $p^b(1-b)^{a-b}$ the same formula is substituted by supposing $m=a-b$, $n=f+h-a+b-1$, whence the term will produce

$$\frac{(v-h+1)(v-h+2)\dots(v+f-a+b-1)(v-f+1)(v-f+2)\dots(2v-f-h+1)}{(v-f+a-b+1)(v-f+a-b+2)\dots(2v)}$$

which is the term of the first case multiplied by

$$\frac{(v-f+1)(v-f+2)\dots(v-f+a-b)}{(v+f-1)(v-f-2)\dots(v+f-a+b)}.$$

But it is $a=f+h-1$, $b=h$, and also this multiplier will be

$$\frac{(v-f+1)(v-f+2)\dots(v-1)}{(v+f-1)(v-f-2)\dots(v+1)}$$

by substituting values mentioned above the sought Probability will be

$$\frac{(v-h+1)(v-h+2)\dots(v+f-1)}{(2v-f-h+2)(2v-f-h+3)\dots(2v)}\Bigg(1+\frac{(f+h-1)}{1}\frac{(v-f+1)}{v+f-1} + \frac{(f+h-1)(f+h-2)}{1.2}\frac{(v-f+1)(v-f+2)}{(v+f-1)(v+f-2)}\dots + \frac{(f+h-1)\dots(h+1)}{1.2.3\dots(f-1)}\frac{(n-f+1)(n-f+2)\dots(n-1)}{(f+n-1)(f+n-2)\dots(n+1)}\Bigg)$$

as the Celebrated Laplace reports.

In the same manner all Problems are able to be solved which the Celebrated Lagrange has treated and which are contained in the preceding dissertation, and in general in the same manner, all Problems of this kind are able to be solved, by preserving the coefficients and whatever terms, and in the place of the terms themselves by substituting the terms which result from the formula demonstrated in §1.

13. The method which we are using is able to be reduced to the method of the Celebrated Laplace the other in an easy manner, it is sufficient to apply the integral calculus in the summation of series. Thus the series

$$\frac{1}{p+1}-\frac{q}{1}\frac{1}{p+2}+\frac{q(q-1)}{1.2}\frac{1}{p+3}-\frac{q(q-1)(q-2)}{1.2.3}\frac{1}{p+4}\dots\pm\frac{1}{p+q+1}$$

is able to take this form

$$\frac{x^{p+1}}{p+1} - \frac{q}{1}\frac{x^{p+2}}{p+2} + \frac{q(q-1)}{1.2}\frac{x^{p+3}}{p+3} - \frac{q(q-1)(q-2)}{1.2.3}\frac{x^{p+4}}{p+4} \text{ etc. } = S$$

here is $S = 0$ where $x = 0$, and after integrating the series there must become $S = 1$. Therefore there is

$$\frac{dS}{dx} = x^p\left[1 - qx + \frac{q(q-1)}{1.2}x^2 - \frac{q(q-1)(q-2)}{1.2.3}x^3 \text{ etc.}\right] = x^p(1-x)^q,$$
$$dS = x^p(1-x)^q dx,$$

$S = x^p(1-x)^q dx$, by being integrated from $x = 0$ to $x = 1$, as the Celebrated Laplace reports.
In the same manner the series

$$\frac{n^q\alpha^{p+1}}{p+1} - qn^{q-1}\frac{\alpha^{p+2}}{p+2} + \frac{q(q-1)}{1.2}n^{q-2}\frac{\alpha^{p+3}}{p+3} \text{ etc.}$$

is able to take this form

$$n^q\alpha^{p+1}\left(\frac{x^{p+1}}{p+1} - \frac{\frac{q\alpha}{n}}{p+2}x^{p+2} + \frac{q(q-1)}{1.2}\frac{\alpha^2}{1.2}\frac{x^{p+3}}{p+3} \text{ etc.}\right)$$

Let

$$\frac{x^{p+1}}{p+1} - \frac{q\alpha}{n}\frac{x^{p+2}}{p+2} + \frac{q(q-1)}{1.2}\frac{\alpha^2}{1.2}\frac{x^{p+3}}{p+3} \text{ etc.} = S',$$

it will produce

$$\frac{dS}{dx} = x^p\left[1 - \frac{q\alpha}{n}x + \frac{q(q-1)}{1.2}\frac{\alpha^2}{n^2}x^2 - \frac{q\ldots(q-2)}{1.2.3}\frac{\alpha^3}{n^3}x^3 \text{ etc.}\right] = x^p\left(1 - \frac{\alpha}{n}x\right)^q,$$

therefore $dS' = x^p\left(1 - \frac{\alpha}{n}x\right)^q dx,\ S' = \int x^p\left(1 - \frac{\alpha}{n}x\right)^q dx$. Now let there be

$$\frac{\alpha}{n} = \frac{p}{p+q} - \theta,$$

it will produce

$$S' = \int x^p\left[1 - \left(\frac{p}{p+q} - \theta\right)x\right]^q dx,$$

by being integrated from $x = 0$ to $x = 1$, and dividing the result by n^{p+q+1}, the series becomes

$$\left(\frac{p}{p+q} - \theta\right)^{p+1}\int x^p\left[1 - \left(\frac{p}{p+q} - \theta\right)x\right]^q dx.$$

But it is, (by being integrated from $x = 0$ to $x = 1$)

$$\int x^p(1-vx)^q dx = \frac{1}{v+1}\int x^p(1-x)^q dx$$

(by being integrated from $x = 1$ to $x = v$), because in both cases it produces the series

$$\frac{1}{p+1} - \frac{qv}{p+2} + \frac{q(q-1)}{1.2}v^2 \text{ etc.}$$

Therefore the series will be

$$\left(\frac{p}{p+q} - \theta\right)^{p+1} \int x^p(1-x)^q dx,$$

by being integrated from $x = 0$ to $x = \frac{p}{p+q} - \theta$. In like manner the calculation will be in other cases.

14. According to the aforesaid it follows the Probability originating from the causes of effects, whatever method to require corresponds to two parts. In the first part the formulas are being assigned which represent this Probability, in the other part they indicate approximations which deliver possible uses of these formulas where enormous numbers are present. We see in the previous § integral formulas used by the men the most celebrated Laplace, Condorcet etc. to be nothing other than abbreviated formulas which coincide with the common method of combinations. In truth $\int x^p(1-x)^q dx$ (by being integrated from $x = 0$ to $x = 1$) is nothing other than the sum of the terms which are produced from the unfolding of the formula $x^p(1-x)^q$, by substituting successively for x of the values $\frac{1}{n}$, $\frac{2}{n}$, $\frac{3}{n}, \ldots$ $\frac{n-1}{n}$ and supposing the quantity n infinite. This conclusion is deduced from the very beginnings themselves of the integrals of the calculus. Demonstrated once by this analogy, the integral formulas are able to be used for the sake of brevity, and chiefly these abridgments are useful while the quantities arrive more composite, for the sake of an example, $x^p(1-x)^q x'^{p'}(1-x')^{q'}$, and the quantities $\frac{1}{n}$, $\frac{2}{n}$, $\frac{3}{n}, \ldots x$ are substituted for x' and the quantities $\frac{1}{n}$, $\frac{2}{n}$, $\frac{3}{n}, \ldots$ $\frac{n-1}{n}$ for x. In this case it will be sufficient to integrate the formula $x^p(1-x)^q x'^{p'}(1-x')^{q'} dx$ from $x' = 0$ to $x' = x$, and from $x = 0$ to $x = 1$. I shall add only one example lest this inquiry become longer.

The Celebrated Laplace supposes to have been born in Paris between certain times p boys q girls, but in London within a different interval of time p' boys q' girls, and he seeks the Probability, the cause which produces boys to be more efficient than in London. From the aforesaid it follows this Probability to be represented by the formula

$$\frac{\iint x^p(1-x)^q x'^{p'}(1-x')^{q'} dx dx'}{\iint x^p(1-x)^q x'^{p'}(1-x')^{q'} dx dx'}$$

(with the numerator being integrated from $x' = 0$ to $x' = x$, and from $x = 0$ to $x = 1$, but the denominator from $x' = 0$ to $x' = 1$, and from $x = 0$ to $x = 1$.) In this first part no difficulty is involved, and by itself its general rule of the calculation of probabilities is being applied for its own sake. However the other part which is concerned with the approximation manner, is subject to enormous difficulty, as it surpassed all things, the Celebrated Laplace has discovered the more sublime Analysis. Now we see in the preceding by what manner the common calculus of series were able to lead to analogous conclusions. In this case the integral of the denominator permits no difficulty. And indeed

$$x'^{p'}(1-x')^{q'} = x'^{p'} - \frac{q'}{1}x'^{p'+1} + \frac{q'(q'-1)}{1.2}x'^{p'+2} - \frac{q'\ldots(q'-2)}{1\ldots3}x'^{p'+3}\cdots x'^{p'+q'}$$

$$\int x'^{p'}(1-x')^{q'}dx' = \frac{x'^{p'+1}}{p'+1} - \frac{q'}{1}\frac{x'^{p'+2}}{p'+2} + \frac{q'(q'-1)}{1.2}\frac{x'^{p'+3}}{p'+3}\cdots \pm \frac{x'^{p'+q'+1}}{p'+q'+1} \text{ (a)}$$

$=$ (by being integrated from $x = 0$ to $x = 1$)

$$\frac{1}{p'+1} - \frac{q'}{1}\frac{1}{p'+2} + \frac{q'(q'-1)}{1.2}\frac{1}{p'+3} - \frac{q'\ldots(q'-2)}{1\ldots3}\frac{1}{p'+4}\cdots \pm \frac{1}{p'+q'+1}.$$

Therefore

$$\iint x^p(1-x)^q x'^{p'}(1-x')^{q'}dxdx' =$$
$$\left(\frac{1}{p'+1} - \frac{q'}{1}\frac{1}{p'+2} + \frac{q'(q'-1)}{1.2}\frac{1}{p'+3} - \frac{q'\ldots(q'-2)}{1\ldots3}\frac{1}{p'+4}\cdots \pm \frac{1}{p'+q'+1}\right)$$
$$\times\left(\frac{1}{p+1} - \frac{q}{1}\frac{1}{p+2} + \frac{q(q-1)}{1.2}\frac{1}{p+3} - \frac{q\ldots(q-2)}{1\ldots3}\frac{1}{p+4}\cdots \mp \frac{1}{p+q+1}\right)$$
$$= \frac{1\ldots q'}{(p'+1)\ldots(p'+q'+1)}\frac{1\ldots q}{(p+1)\ldots(p+q+1)}.$$

However the integral of the higher numerator is explored as seen from the following. I return to formula (a) which is made, supposing $x = x'$,

$$\int x'^{p'}(1-x')^{q'}dx' = \frac{x^{p'+1}}{p'+1} - \frac{q'}{1}\frac{x^{p'+2}}{p'+2} + \frac{q'(q'-1)}{1.2}\frac{x^{p'+3}}{p+3}\cdots \pm \frac{x^{p'+q'+1}p'+}{p'+q'+1}.$$

Therefore

$$\iint x^p(1-x)^q x'^{p'}(1-x')^{q'}dxdx' =$$
$$\int\left(\frac{x^{p+p'+1}}{p'+1} - \frac{q'}{1}\frac{x^{p+p'+2}}{p'+2} + \frac{q'(q'-1)}{1.2}\frac{x^{p+p'+3}}{p'+3}\cdots \pm \frac{x^{p+p'+q'+1}}{p'+q'+1}\right)$$
$$\times\left(1 - \frac{q}{1}x + \frac{q(q-1)}{1.2}x^2 - \frac{q\ldots(q-2)}{1\ldots3}x^3\cdots \pm x^q\right) =$$

(by integrating from $x=0$ to $x=1$)

$$\frac{1}{p'+1}\left(\frac{1}{p+p'+2}-\frac{q}{p+p'+3}+\frac{q(q-1)}{1.2}\frac{1}{p+p'+4}\cdots\pm\frac{1}{p+p'+q+2}\right)$$
$$-\frac{q'}{p'+3}\left(\frac{1}{p+p'+3}-\frac{q}{p+p'+4}+\frac{q(q-1)}{1.2}\frac{1}{p+p'+5}\cdots\pm\frac{1}{p+p'+q+3}\right)$$
$$+\frac{\frac{q'(q'-1)}{1.2}}{p'+3}\left(\frac{1}{p+p'+4}-\frac{q}{p+p'+5}+\frac{q(q-1)}{1.2}\frac{1}{p+p'+6}\cdots\pm\frac{1}{p+p'+q+4}\right)$$
$$-\frac{\frac{q'\ldots(q'-2)}{1\ldots3}}{p'+4}\left(\frac{1}{p+p'+5}-\frac{q}{p+p'+6}+\frac{q(q-1)}{1.2}\frac{1}{p+p'+7}\cdots\pm\frac{1}{p+p'+q+5}\right)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\pm\frac{1}{p'+q'+1}\left(\frac{1}{p+p'+q'+2}-\frac{q}{p+p'+q'+3}+\frac{q(q-1)}{1.2}\frac{1}{p+p'+q'+4}\right.$$
$$\left.\cdots\pm\frac{1}{p+p'q+q'+2}\right)=$$

(by summing this series by the method of §1.)

$$\frac{1}{p'+1}\left(\frac{1.2\ldots q}{(p+p'+2)\ldots(p+p'+q+2)}-\frac{q'}{p'+2}\frac{1.2\ldots q}{(p+p'+3)\ldots(p+p'+q+3)}\right.$$
$$+\frac{\frac{q'(q'-1)}{1.2}}{p'+3}\frac{1.2\ldots q}{(p+p'+4)\ldots(p+p'+q+4)}-\frac{\frac{q'\ldots(q'-2)}{1\ldots3}}{p'+4}\frac{1.2\ldots q}{(p+p'+5)\ldots(p+p'+q+5)}$$
$$\cdots+\frac{1}{p'+q'+1}\frac{1.2\ldots q}{(p+p'+q'+2)\ldots(p+p'q+q'+2)}=$$
$$\frac{1.2\ldots q}{(p+p'+2)\ldots(p+p'+q+2)}\left(\frac{1}{p'+1}-\frac{q'}{p'+2}\frac{(p+p'+2)}{p+p'+q+3}\right.$$
$$+\frac{\frac{q'(q'-1)}{1.2}}{p'+3}\frac{(p+p'+2)(p+p'+3)}{(p+p'+q+3)(p+p'+q+4)}-\frac{\frac{q'\ldots(q'-2)}{1\ldots3}}{p'+4}\frac{(p+p'+2)\ldots(p+p'+4)}{(p+p'+q+3)\ldots(p+p'+q+5)}$$
$$\left.\cdots\pm\frac{1}{p'+q'+1}\frac{(p+p'+2)\ldots(p+p'+q'+1)}{(p+p'+q+3)\ldots(p+p'+q+q'+2)}\right)$$

The series

$$\frac{1}{p'+2}-\frac{q'}{p'+2}\frac{(p+p'+2)}{p+p'+q+3}\text{ etc.}$$

provides, by substituting successively for q' the values $0,\ 1,\ 2,\ 3$ etc.

$$\begin{array}{r|l}
q=0 & \frac{1}{p'+1} \\
q=1 & \frac{1}{(p'+1)(p'+2)}+\frac{q+1}{(p'+2)(p+p'+q+3)} \\
q=2 & \frac{1.2}{(p+1)\ldots(p'+3)}+\frac{2(q+1)}{(p'+2)(p'+3)(p+p'+q+3)}+\frac{(q+1)(q+2)}{(p'+3)(p+p'+q+3)(p+p'+q+4)} \\
q=3 & \frac{1.2.3}{(p+1)\ldots(p'+4)}+\frac{3.2(q+1)}{(p'+2)\ldots(p'+4)(p+p'+q+3)} \\
 & +\frac{3(q+1)(q+2)}{(p'+3)(p'+4)(p+p'+q+3)(p+p'+q+4)}+\frac{(q+1)(q+2)(q+3)}{(p'+4)(p+p'+q+3)(p+p'+q+4)}
\end{array}$$

Now it is clear by analogy, and hence there exists for

$$q' = q' \qquad \frac{1.2\ldots q'}{(p'+1)\ldots(p'+q'+1)} + \frac{2.3\ldots q'(q+1)}{(p'+2)\ldots(p'+q'+1)(p+p'+q+3)}$$
$$+ \frac{3.4\ldots q'(q+1)(q+2)}{(p'+3)\ldots(p'+q'+1)(p+p'+q+3)(p+p'+q+4)}$$
$$+ \frac{4.5\ldots q'(q+1)\ldots(q+3)}{(p'+4)\ldots(p'+q'+1)(p+p'+q+3)\ldots(p+p'+q+5)}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ \frac{(q'-2)(q'-1)q'(q+1)\ldots(q+q'-3)}{(p'+q'-2)\ldots(p'+q'+1)(p+p'+q+3)\ldots(p+p'+q+q'-1)}$$
$$+ \frac{(q'-1)q'(q+1)\ldots(q+q'-2)}{(p'+q'-1)\ldots(p'+q'+1)(p+p'+q+3)\ldots(p+p'+q+q')}$$
$$+ \frac{q'(q+1)\ldots(q+q'-1)}{(p'+q')\ldots(p'+q'+1)(p+p'+q+3)\ldots(p+p'+q+q'+1)}$$
$$+ \frac{(q+1)\ldots(q+q')}{(p'+q'+1)(p+p'+q+3)\ldots(p+p'+q+q'+2)} =$$

$$\frac{(q+1)\ldots(q+q')}{(p+p'+q+3)\ldots(p+p'+q+q'+2)}\left(\frac{1}{p'+q'+1} + \frac{q'}{q+q'}\frac{(p+p'+q+q'+2)}{(p'+q'+1)(p'+q')}\right.$$
$$+ \frac{q'(q'-1)}{(q+q')(q+q'-1)}\frac{(p+p'+q+q'+2)(p+p'+q+q'+1)}{(p'+q'+1)(p'+q')(p'+q'-1)}$$
$$+ \frac{q'(q'-1)(q'-2)}{(q+q')(q+q'-1)(q+q'-2)}\frac{(p+p'+q+q'+2)\ldots(p+p'+q+q')}{(p'+q'+1)(p'+q')\ldots(p'+q'-2)}$$
$$+ \frac{q'(q'-1)\ldots(q'-3)}{(q+q')(q+q'-3)}\frac{(p+p'+q+q'+2)\ldots(p+p'+q+q'-1)}{(p'+q'+1)\ldots(p'+q'-3)}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$+ \frac{q'\ldots 1}{(q+q')(q+1)}\frac{(p+p'+q+q'+2)\ldots(p+p'+q+3)}{(p'+q'+1)\ldots(p'+1)} =$$

(by employing the first for approximation)

$$\frac{(q+1)\ldots(q+q')}{(p+p'+q+3)\ldots(p+p'+q+q'+2)}$$
$$\left(\frac{1}{p'+q'} + \frac{q'}{q+q'}\ \frac{(p+p'+q+q')}{(p'+q')(p'+q')}\right.$$
$$\left. + \ \frac{q'^2}{(q+q')^2}\frac{(p+p'+q+q')^2}{(p'+q')(p'+q')^2}\text{ etc.}\right) =$$
$$\frac{(q+1)\ldots(q+q')}{(p+p'+q+3)\ldots(p+p'+q+q'+1)}\frac{1}{(p'+q') - q'\frac{(p+p'+a+q')}{q+q'}} =$$
$$\frac{(q+1)\ldots(q+q')}{(p+p'+q+3)\ldots(p+p'+q+q'+2)}\frac{(q+q')}{p'q-pq'}.$$

By multiplying by

$$\frac{1.2\ldots q}{(p+p'+2)\ldots(p+p'+q+2)}$$

the complete integral will appear, $=$

$$\frac{1\ldots(q+q')}{(p+p'+2)\ldots(p+p'+q+q'+2)}\frac{(q+q')}{p'q-pq'} = \\ \frac{\sqrt{2\pi}(p+p')^{p+p'+\frac{3}{2}}(q+q')^{q+q'+\frac{3}{2}}}{(p'q-pq')(p+p'+q+q')^{p+p'+q+q'+\frac{1}{2}}}.$$

If it be necessary to progress further, let for the sake of brevity,

$$\alpha=(p'+q'+1), \quad \beta=q', \quad \gamma=q'+q, \quad \delta=p+p'+q+q'+2,$$

whence it will produce the series

$$\frac{1}{\alpha}+\frac{\beta}{\gamma}\frac{\delta}{\alpha(\alpha-1)}+\frac{\beta(\beta-1)}{\gamma(\gamma-1)}\frac{\delta(\delta-1)}{\alpha(\alpha-1)(\alpha-2)} \\ +\frac{\beta(\beta-1)(\beta-2)}{\gamma(\gamma-1)(\gamma-2)}\frac{\delta(\delta-1)(\delta-2)}{\alpha(\alpha-1)(\alpha-2)(\alpha-3)} \\ +\frac{\beta(\beta-1)(\beta-2)(\beta-3)\delta(\delta-1)(\delta-2)(\delta-3)}{\gamma(\gamma-1)(\gamma-2)(\gamma-3)\alpha(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)} \text{ etc. } =$$

$$\frac{1}{\alpha}+\frac{\beta}{\gamma}\frac{\delta}{\alpha}\left(\frac{1}{\alpha}+\frac{1}{\alpha^2}+\frac{1}{\alpha^3}\right) \\ +\frac{\beta}{\gamma}(\beta-1)\frac{\delta}{\alpha}(\delta-1)\left(\frac{1}{\gamma}+\frac{1}{\gamma^2}+\frac{1}{\gamma^3}\right)\left(\frac{1}{\alpha^2}+\frac{3}{\alpha^3}+\frac{7}{\alpha^4}\right) \\ +\frac{\beta}{\gamma}(\beta\beta-3\beta+2)\frac{\delta}{\alpha}(\delta\delta-3\delta+2)\left(\frac{1}{\alpha^3}+\frac{6}{\alpha^4}+\frac{25}{\alpha^5}\right)\left(\frac{1}{\gamma^2}+\frac{3}{\gamma^3}+\frac{7}{\gamma^4}\right) \\ +\frac{\beta}{\gamma}(\beta^3-6\beta^2+11\beta)\left(\frac{1}{\gamma^3}+\frac{6}{\gamma^4}+\frac{25}{\gamma^5}\right)(\delta^3-6\delta^2+11\delta)\left(\frac{1}{\alpha^4}+\frac{10}{\alpha^5}+\frac{65}{\alpha^6}\right) \text{ etc.} =$$

$$\frac{1}{\alpha}+\frac{\beta\delta}{\alpha^2\gamma}+\frac{\beta^2\delta^2}{\alpha^3\gamma^2}+\frac{\beta^3\delta^3}{\alpha^4\gamma^3}+\frac{\beta^4\delta^4}{\alpha^5\gamma^4}\text{ etc.}$$

$$-\beta\delta\left(\frac{\beta+\delta-\gamma)}{\alpha^3\gamma^2}+\frac{\beta\delta(3\beta+3\delta-\alpha-3\gamma)}{\alpha^4\gamma^3}+\frac{\beta^2\delta^2(6\beta+6\delta-3\alpha-6\gamma)}{\alpha^5\gamma^4}\right.$$

$$\left.+\ \frac{\beta^3\delta^3(10\beta+10\delta-6\alpha-10\gamma)}{\alpha^6\gamma^5}\text{ etc.}\right)$$

$$+\frac{\beta\delta(\alpha\gamma+\gamma^2)}{\alpha^4\gamma^3}-\frac{\beta\delta(\beta+\delta)(\alpha+3\delta)}{\alpha^4\gamma^3}+\frac{\beta\delta(2\beta^2+9\beta\delta+2\delta^2)}{\alpha^4\gamma^3}$$

$$+\frac{\beta^2\delta^2(\alpha^2+3\alpha\gamma+7\gamma^2)}{\alpha^5\gamma^4}-\frac{3\beta^2\delta^2(\beta+\delta)(3\alpha+6\gamma)}{\alpha^5\gamma^4}$$

$$+\frac{\beta^2\delta^2(11\beta^2+36\beta\delta+11\delta^2)}{\alpha^5\gamma^4}$$

$$+\frac{\beta^3\delta^3(7\alpha^2+18\alpha\gamma+25\gamma^2)}{\alpha^6\gamma^5}-\frac{6\beta^3\delta^3(\beta+\delta)(6\alpha+10\gamma)}{\alpha^6\gamma^5}$$

$$+\frac{\beta^3\delta^3(35\beta^2+100\beta\delta+35\delta^2)}{\alpha^6\gamma^5}$$

$$+\frac{\beta^4\delta^4(25\alpha^2+60\alpha\gamma+65\gamma^2)}{\alpha^7\gamma^6}-\frac{10\beta^4\delta^4(\beta+\delta)(10\alpha+15\gamma)}{\alpha^7\gamma^6}$$

$$+\frac{\beta^4\delta^4(85\beta^2+225\beta\delta+85\delta^2)}{\alpha^7\gamma^6}\text{ etc.}$$

The sum of the first series is =

$$\frac{1}{\alpha-\frac{\beta\delta}{\gamma}}=\frac{\gamma}{\alpha\gamma-\beta\delta}=\frac{q'+q}{p'q-pq'+q+q'}.$$

The other series has taken this form

$$-\left(\frac{1}{\alpha^3}+\frac{3\beta\delta}{\alpha^4\gamma}+\frac{6\beta^2\delta^2}{\alpha^5\gamma^2}+\frac{10\beta^3\delta^3}{\alpha^6\gamma^3}\text{ etc.}\right)\beta\delta\left(\frac{\beta+\delta}{\gamma^2}-\frac{1}{\gamma}-\frac{\beta\delta}{\gamma^3}\right)=$$

$$-\frac{\beta\delta}{\gamma^3}[(\beta+\delta)\gamma-\beta\delta-\gamma^2]\frac{1}{\left(\alpha-\frac{\beta\delta}{\gamma}\right)^3}=-\frac{\beta\delta(\beta+\delta)\gamma-\beta\delta-\gamma^2}{(\alpha\gamma-\beta\delta)^3}=$$

$$-\frac{q'(p+p'+q+q'+2)(p+p'+2)q}{(p'q-pq'+q+q')^3}.$$

The third series has taken this form

$$\frac{\beta\delta}{\gamma^3}\left(\frac{\gamma^2}{\alpha^4}+\frac{\beta\delta(\alpha^2+3\alpha\gamma+7\gamma^2)}{\alpha^5\gamma}+\frac{\beta^2\delta^2(7\alpha^2+18\alpha\gamma+25\gamma^2)}{\alpha^6\gamma^2}\right.$$
$$\left.+\ \frac{\beta^3\delta^3(25\alpha^2+60\alpha\gamma+65\gamma^2)}{\alpha^7\gamma^3}\text{ etc.}\right)$$
$$-\ \frac{\beta\delta(\beta+\delta)}{\gamma^3}\left(\frac{\alpha+3\gamma}{\alpha^4}+\frac{3\beta\delta(3\alpha+6\gamma)}{\alpha^5\gamma}+\frac{6\beta^2\delta^2(6\alpha+10\gamma)}{\alpha^6\gamma^2}\right.$$
$$\left.+\ \frac{10\beta^3\delta^3(10\alpha+15\gamma)}{\alpha^7\gamma^3}\text{ etc.}\right)$$
$$+\ \frac{\beta\delta}{\gamma^3}\left(\frac{2\beta^2+9\beta\delta+2\delta^2}{\alpha^4}+\frac{\beta\delta(11\beta^2+36\beta\delta+11\delta^2)}{\alpha^5\gamma}\right.$$
$$\left.+\ \frac{\beta^2\delta^2(35\beta^2+100\beta\delta+35\delta^2)}{\alpha^6\gamma^2}+\frac{\beta^3\delta^3(85\beta^2+225\beta\delta+85\delta^2)}{\alpha^7\gamma^2}\text{ etc.}\right)$$

This series is able to be resolved in the following manner,

$$\frac{\beta^2\delta^2\alpha^2}{\gamma^4}\left(\frac{1}{\alpha^5}+\frac{7\beta\delta}{\alpha^6\gamma}+\frac{25\beta^2\delta^2}{\alpha^7\gamma^2}+\frac{65\beta^3\delta^3}{\alpha^8\gamma^3}+\frac{140\beta^4\delta^4}{\alpha^9\gamma^4}+\frac{266\beta^5\delta^5}{\alpha^{10}\gamma^5}\text{ etc.}\right)$$

However of the numbers

$$\begin{array}{ccccccccccc}
1 & & 7 & & 25 & & 65 & & 140 & & 266\\
& 6 & & 18 & & 40 & & 75 & & 126 & \\
& & 12 & & 22 & & 35 & & 51 & & \\
& & & 10 & & 13 & & 16 & & & \\
& & & & 3 & & 3 & & & &
\end{array}$$

the fourth differences are constants, therefore by employing the method exposed in the § above, the denominator of the generating fraction of the series will be

$$=\left(\alpha-\frac{\beta\delta}{\gamma}\right)^{-\delta}=\begin{array}{llll}
\frac{1}{\alpha^5} & +\frac{5\beta\delta}{\alpha^6\gamma} & +\frac{15\beta^2\delta^2}{\alpha^7\gamma^2} & +\frac{35\beta^3\delta^3}{\alpha^8\gamma^3}\\
1 & +\frac{B\beta\delta}{\alpha\gamma} & & \\
\hline
\frac{1}{\alpha^5} & +\frac{5\beta\delta}{\alpha^6\gamma} & +\frac{15\beta^2\delta^2}{\alpha^7\gamma^2} & +\frac{35\beta^3\delta^3}{\alpha^8\gamma^3}\\
& +B & +5B & +15B
\end{array}$$

I have multiplied this series by $1+\frac{B\beta\delta}{\alpha\gamma}$, (hence B being constant term) and by comparing terms it produces $B+5=7$, therefore $B=2$, and thus it satisfies the other conditions. Therefore there exists a part of the series=

$$\frac{\alpha^2\beta^2\delta^2}{\gamma^4}\frac{\left(\alpha+\frac{2\beta\delta}{\alpha\gamma}\right)}{\left(\alpha-\frac{\beta\delta}{\gamma}\right)^5}=\frac{\alpha\beta^2\delta^2(\alpha\gamma+2\beta\delta)}{(\alpha\gamma-\beta\delta)^5}.$$

The sequence of members is

$$\frac{3\alpha\beta^2\delta^2}{\gamma^3}\left(\frac{1}{\alpha^5}+\frac{6\beta\delta}{\alpha^6\gamma}+\frac{20\beta^2\delta^2}{\alpha^7\gamma^2}+\frac{50\beta^3\delta^3}{\alpha^8\gamma^3}+\frac{105\beta^4\delta^4}{\alpha^9\gamma^4}+\frac{196\beta^5\delta^5}{\alpha^{10}\gamma^5}\text{ etc.}\right)$$

However of the numbers

$$\begin{array}{ccccccccccc}
1 & & 6 & & 20 & & 50 & & 105 & & 196 \\
& 5 & & 14 & & 30 & & 55 & & 91 & \\
& & 9 & & 16 & & 25 & & 36 & & \\
& & & 7 & & 9 & & 11 & & & \\
& & & & 2 & & 2 & & & &
\end{array}$$

the fourth differences are constants, likewise therefore I enjoy by the method

$$\begin{array}{llll}
\frac{1}{\alpha^5} & +\frac{5\beta\delta}{\alpha^6\gamma} & +\frac{15\beta^2\delta^2}{\alpha^7\gamma^2} & +\frac{35\beta^3\delta^3}{\alpha^8\gamma^3}\text{ etc.} \\
1 & +\frac{B\beta\delta}{\alpha\gamma} & & \\
\hline
\frac{1}{\alpha^5} & +\frac{5\beta\delta}{\alpha^6\gamma} & +\frac{15\beta^2\delta^2}{\alpha^7\gamma^2} & +\frac{35\beta^3\delta^3}{\alpha^8\gamma^3}\text{ etc.} \\
& +B & +5B & +15B
\end{array}$$

it produces $B = 1$, therefore this member is $=$

$$\frac{3\alpha\beta^2\delta^2}{\gamma^3}\frac{\left(\alpha+\frac{\beta\delta}{\alpha\gamma}\right)}{\left(\alpha-\frac{\beta\delta}{\gamma}\right)^5} = \frac{3\beta^2\delta^2\gamma(\alpha\gamma+\beta\delta)}{(\alpha\gamma-\beta\delta)^5}.$$

The sequence of members is

$$\frac{\alpha\beta\delta}{\gamma}\left(\frac{1}{\alpha^5}+\frac{7\beta\delta}{\alpha^6\gamma}+\frac{25\beta^2\delta^2}{\alpha^7\gamma^2}+\frac{65\beta^3\delta^3}{\alpha^8\gamma^3}\text{ etc.}\right) = \frac{\beta\delta\gamma^3(\alpha\gamma+2\beta\delta)}{(\alpha\gamma-\beta\delta)^5}.$$

The sequence of members is

$$-\frac{\alpha^2\beta\delta(\beta+\delta)}{\gamma^3}\left(\frac{1}{\alpha^5}+\frac{9\beta\delta}{\alpha^6\gamma}+\frac{36\beta^2\delta^2}{\alpha^7\gamma^2}+\frac{100\beta^3\delta^3}{\alpha^8\gamma^3}+\frac{225\beta^4\delta^4}{\alpha^9\gamma^4}\text{ etc.}\right)$$

However of the numbers

$$\begin{array}{ccccccccccccc}
1 & & 9 & & 36 & & 100 & & 225 & & 441 & & 784 \\
& 8 & & 27 & & 64 & & 125 & & 216 & & 343 & \\
& & 19 & & 37 & & 61 & & 91 & & 123 & & \\
& & & 6 & & 6 & & 6 & & 6 & & &
\end{array}$$

the fourth differences are constants, likewise I enjoy by the method

$$\begin{array}{llll}
\frac{1}{\alpha^5} & +\frac{5\beta\delta}{\alpha^6\gamma} & +\frac{15\beta^2\delta^2}{\alpha^7\gamma^2} & +\frac{35\beta^3\delta^3}{\alpha^8\gamma^3}\text{ etc.} \\
1 & +\frac{B\beta\delta}{\alpha\gamma} & +\frac{C\beta^2\delta^2}{\alpha^2\gamma^2} & \\
\hline
\frac{1}{\alpha^5} & +\frac{5\beta\delta}{\alpha^6\gamma} & +\frac{15\beta^2\delta^2}{\alpha^7\gamma^2} & +\frac{35\beta^3\delta^3}{\alpha^8\gamma^3}\text{ etc.} \\
& +B & +5B & +15B \\
& & +C & +5C
\end{array}$$

it produces $B = 4$, $C = 1$, therefore this member becomes $=$

$$\frac{-\alpha^2\beta\delta(\beta+\delta)}{\gamma^3}\frac{\left(1+\frac{4\beta\delta}{\alpha\gamma}+\frac{\beta^2\delta^2}{\alpha^2\gamma^2}\right)}{\left(\alpha-\frac{\beta\delta}{\gamma}\right)^5} = \frac{-\beta\delta(\beta+\delta)(\alpha^2\gamma^2+4\alpha\beta\gamma\delta+\beta^2\delta^2)}{(\alpha\gamma-\beta\delta)^5}$$

The sequence of members is

$$\frac{-3\alpha\beta\delta(\beta+\delta)}{\gamma^2}\left(\frac{1}{\alpha^5}+\frac{6\beta\delta}{\alpha^6\gamma}+\frac{20\beta^2\delta^2}{\alpha^7\gamma^2}+\frac{50\beta^3\delta^3}{\alpha^8\gamma^3}\text{ etc.}\right)=$$

$$\frac{-3\alpha\beta\delta(\beta+\delta)}{\gamma^2}\frac{\left(\alpha+\frac{\beta\delta}{\alpha\gamma}\right)}{\left(\alpha-\frac{\beta\delta}{\gamma}\right)^5}=\frac{-3\beta\gamma^2\delta(\beta+\delta)(\alpha\gamma+\beta\delta)}{(\alpha\gamma-\beta\delta)^5}$$

The sequence of members is

$$\frac{2\alpha\beta^3\delta}{\gamma}\left(\frac{1}{\alpha^5}+\frac{\frac{11}{2}\beta\delta}{\alpha^6\gamma}+\frac{\frac{35}{2}\beta^2\delta^2}{\alpha^7\gamma^2}+\frac{\frac{85}{2}\beta^3\delta^3}{\alpha^8\gamma^3}\text{ etc.}\right)$$

However of the numbers 2, 11, 35, 85 etc. the fourth differences are constants, likewise I enjoy by the method,

$$\begin{array}{llll}
\frac{1}{\alpha^5} & +\frac{5\beta\delta}{\alpha^6\gamma} & +\frac{15\beta^2\delta^2}{\alpha^7\gamma^2} & +\frac{35\beta^3\delta^3}{\alpha^8\gamma^3}\text{ etc.}\\
1 & +\frac{B\beta\delta}{\alpha\gamma} & & \\
\frac{1}{\alpha^5} & +\frac{5\beta\delta}{\alpha^6\gamma} & +\frac{15\beta^2\delta^2}{\alpha^7\gamma^2} & +\frac{35\beta^3\delta^3}{\alpha^8\gamma^3}\text{ etc.}\\
 & +B & +5B & +15B
\end{array}$$

it produces $B=\frac{1}{2}$, therefore it happens this member =

$$\frac{2\alpha\beta^3\delta}{\gamma^3}\frac{\left(\alpha+\frac{\beta\delta}{2\alpha\gamma}\right)}{\left(\alpha-\frac{\beta\delta}{\gamma}\right)^5}=\frac{\beta^3\gamma\delta(2\alpha\gamma+\beta\delta)}{(\alpha\gamma-\beta\delta)^5}.$$

The sequence of members is

$$\frac{\alpha^2\beta\delta}{\gamma^2}\left(\frac{1}{\alpha^5}+\frac{9\beta\delta}{\alpha^6\gamma}+\frac{36\beta^2\delta^2}{\alpha^7\gamma^2}+\frac{100\beta^3\delta^3}{\alpha^8\gamma^3}\text{ etc.}\right)=$$

$$\frac{\alpha^2\beta\delta}{\gamma^2}\frac{\left(\alpha+\frac{4\beta\delta}{\alpha\gamma}+\frac{\beta^2\delta^2}{\alpha^2\gamma^2}\right)}{\left(\alpha-\frac{\beta\delta}{\gamma}\right)^5}=\frac{\beta\gamma\delta(\alpha^2\gamma^2+4\alpha\beta\gamma\delta+\beta^2\delta^2)}{(\alpha\gamma-\beta\delta)^5}$$

The last member is

$$\frac{2\alpha\beta\delta^3}{\gamma^3}\left(\frac{1}{\alpha^5}+\frac{\frac{11}{2}\beta\delta}{\alpha^6\gamma}+\frac{\frac{35}{2}\beta^2\delta^2}{\alpha^7\gamma^2}\text{ etc.}\right)=$$

$$\frac{\alpha^2\beta\delta^3}{\gamma^2}\frac{\left(\alpha+\frac{\beta\delta}{\alpha\gamma}\right)}{\left(\alpha-\frac{\beta\delta}{\gamma}\right)^5}=\frac{\beta\gamma\delta^3(2\alpha\gamma+\beta\delta)}{(\alpha\gamma-\beta\delta)^5}$$

The sum of these values, [by supposing for the sake of brevity

$$T = (\alpha\beta\delta + \gamma^3)(\alpha\gamma + 2\beta\delta) + (3\beta\gamma\delta - \beta\gamma^2 - \gamma^2\delta)(\alpha\gamma + \beta\delta)$$
$$- (\beta + \delta - \gamma)(\alpha^2\gamma^2 + 4\alpha\beta\gamma\delta + \beta^2\delta^2) + (\beta^2\gamma + \gamma\delta^2)(\alpha\gamma + \beta\delta)]$$

is $= \frac{\beta\delta T}{(\alpha\gamma-\beta\delta)^5}$. The integral of the numerator will be therefore =

$$\frac{\gamma}{\alpha\gamma - \beta\delta}\left(1 - \frac{\beta\delta(\beta\gamma + \gamma\delta - \beta\delta - \gamma^2)}{\gamma(\alpha\gamma - \beta\delta)^2} + \frac{\beta\delta T}{\gamma(\alpha\gamma - \beta\delta)^4} \text{ etc.}\right)$$

Likewise it will be permitted to proceed by the method of another. This series, which is similar to the series which the Celebrated Laplace reports, requires many considerations, which here, lest this dissertation be longer, I am forced to omit; it is sufficient here to indicate the method. Still other worthy examples to note may be given which, please God, I will develop in a following dissertation.