RECHERCHES
sur la mortalité de la petite vérole*

Jean Trembley

Mémoires de l'Académie Royale des Sciences et Belles-Lettres... Berlin
1796 (1799) Class de math. pp. 17–38†

§ 1. One of the objects of political arithmetic which has the greatest right to the attention of philosophers, is without doubt that which depends upon the mortality of smallpox. Since inoculation has commenced to be introduced in Europe, the interest that it has excited, & the disputes that it has occasioned, have given birth to the desire to understand in detail the nature of the ravages produced by this cruel malady. We have discussed a long time without solving the matter, because we had few exact observations, which alone would have served as foundation to these researches. One of the greatest geometers of the century, which person has no equal in the art to adapt analysis to the laws of nature, Mr. Daniel Bernoulli has given in 1760 in the *Mémoires de l'Académie des Sciences de Paris*, an Essay on the analysis of the mortality caused by smallpox, & the elegance of this analysis causes admiration of the wisdom of its inventor. I will not report at all this solution which is rather known: I will content myself to make some reflections on the goal that Mr. Bernoulli himself has proposed & on the means that he has employed; I will come to it next in the researches on practice which are properly the object of this memoir.

§ 2. The advantages that one can retrieve from the application of the analysis of natural laws, consists not in the discovery of new effects, of new properties of sensible beings; mathematics supposes necessarily some previous observations, some determined phenomena: but these phenomena one time known with precision, one can deduce from them by the aid of analysis, some laws which did not present themselves at first glance, some hidden relations which give some clearer ideas of the march of nature & of its operations. The geometer must therefore, before all, if he does not wish to believe some chimeras, choose some assumptions based on facts. This is thence an essential condition of all the physico-mathematical sciences, & if these assumptions are precarious or imperfect, the sciences which will result from them will have the same faults. But the effects which one observes in nature having linked among them, one finds sometimes much advantage to depart from one fact rather than from anything else, either that this fact is easier to observe, or that it leads from a simpler & more immediate manner to the consequences that one has proposed to discover. In the question

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. December 22, 2009

†"Researches on the mortality of smallpox." Read to the Academy on 23 June 1796.

of the mortality of smallpox there are two principal effects which one must necessarily deduce from observation, either medially or immediately. The question is to know: 1° What is at each age the fraction which expresses the danger to contract smallpox, by taking for unity the number of persons of this age who have not had it; or in other terms of it: what is at each age the ratio of the number of those who have not yet had it. 2°. What is at each age the fraction which expresses the danger to die of smallpox; by taking for unity the number of persons of that age who have contracted smallpox, or in other terms of it: what is at each age the ratio of the number of those who contract it. These two ratios being determined, it is clear that the danger to die of smallpox at each age is by reason composed of the danger to contract smallpox, & of the one to die if one takes it. If therefore $\frac{1}{n}$ expresses the danger to contract smallpox & $\frac{1}{m}$ the danger to die if one contracts it, the danger to die of smallpox will be expressed by $\frac{1}{mn}$. These two fundamental points could be deduced immediately from experience, if one had some tables which indicated for each age the number of persons who contract smallpox & the number of those who die of it. By combining these tables with the general tables of mortality which teach us how many out of a given number there die of persons at each age, one would arrive to the exact knowledge of these two essential elements. And then the work of the mathematician would consist in deducing thence for each age the state of human kind relative to smallpox, & to determine out of a given number of births the number of persons who contract it & who die of it, & the ratio of the mortality of smallpox to the mortality which results from other maladies. It is true that one can not legitimately hope to have some tables of this perfection, because the number of persons who contract smallpox is very difficult, if not to say impossible, to determine. Instead of this determination, if one had for each age the number of persons who die of other maladies, one would know thence the value of the fraction $\frac{1}{mn}$, & by proceeding with caution one would arrive to determine which enough probability the particular values of $\frac{1}{m}$ & $\frac{1}{n}$, as I will demonstrate below.

§ 3. Mr. Bernoulli supposes that the fractions $\frac{1}{m}$ & $\frac{1}{n}$ remain constants at each age. By means of this assumption, he obtains by employing the infinitesimal calculus, a differential equation of the first degree in three variables, which is found susceptible to integration, this which is rare in the equations of this kind, when they have not been fabricated by design. After having integrated & determined appropriately the constant, he arrives to a formula which gives to him for each age out of a given number of births, the number of persons who have not yet had smallpox, by departing from the table of mortality of Halley, which according to the mortuary records of Breslau gives for each age the number of persons who remain alive out of a given number of births, & by supposing $\frac{1}{n} = \frac{1}{m} = \frac{1}{8}$, that is to say by supposing that at each age the number of persons who contract smallpox is to the number of persons who have not yet had it, as $1 : 8$ & that the number of persons who die of smallpox is to the number of persons who contract it, as $1 : 8$. He deduces thence for each age the ratio of the mortality of smallpox to the mortality caused by other maladies, & there results from his calculations, that the entire mortality of smallpox is around $\frac{1}{13}$ of the total mortality, this which accords rather well with the observations that one has been able to gather until here.

§ 4. Mr. d'Alembert in the second volume of his *Opuscules Mathématiques*, is himself raised with much force against the theory of which I just spoke. He has objected with reason, that it was not at all probable that the fractions $\frac{1}{n}$ & $\frac{1}{m}$ were constants for each age; he has made err some consequences a little difficult on this theory, such as this here, that in the 9th year the number of persons dead of smallpox is to the number of persons dead of other maladies, as $2 : 3$, this which appears a little great. But as this great geometer has combined with his objections some assertions on the calculus of probabilities which have not obtained the suffrage of the mathematicians, & as he has substituted for the elegant analysis of Mr. Bernoulli a mathematical theory so very mathematical, that neither he nor a person that I know, has made the application of it, one has had little regard to his demands, & the theory of Mr. Bernoulli has continued to occupy the geometers. Mr. Lambert in Vol. III of his *Beyträge*[1] has given some researches on this matter, based in great part on this same theory of which however he does not hide the faults. There is joined a multitude of fine & ingenious remarks which characterize all the writings of this celebrated philosopher, but he has not tried to treat the problem other than Mr. Bernoulli.

§ 5. After having examined with attention the analysis of Mr. Bernoulli, I have sought first if the success of the integral calculus was here of an indispensable necessity, & if it would not be possible to arrive to the solution of the problem without exiting from ordinary analysis. Having succeeded in this preliminary research, I wished to see if the formulas which have led me to the end, would not be able to serve in the case where $\frac{1}{m}$ & $\frac{1}{n}$ would not be constants, but would vary from year to year. I have found that the thing was possible, & that consequently these formulas were a great advantage over the infinitesimal analysis of Mr. Bernoulli, in which on is obliged to suppose $\frac{1}{m}$ & $\frac{1}{n}$ constants. Finally I have sought if of the tables which exist, there would not be some one which could serve to give at least an idea of the application of this analysis in practice, & of the nature of the consequences which would result from exact & complete tables of the mortality of smallpox for a great number of years.

§ 5. I will give first the analysis which has led me to the solution of the problem of Mr. Bernoulli, & which is based on the same principles. Let $a^{(0)}$ be a given number of births, & $a^{(1)}$, $a^{(2)}$, $a^{(3)}$, $\ldots a^{(v)}$ the numbers of persons who subsist at the end of $1, 2, 3 \ldots v$ years. We will consider first the first year. Since at the end of one year there survive $a^{(1)}$ persons, there are dead the first year $a^{(0)} - a^{(1)}$ persons; now under the assumption, there have been $\frac{a^{(0)}}{n}$ persons attacked by smallpox, & of this number of persons there are $\frac{a^{(0)}}{mn}$ dead of them, thus there are dead the first year of other maladies $a^{(0)} - a^{(1)} - \frac{a^{(0)}}{mn}$. If the other maladies had carried away no person, at the beginning of the second year, the number of persons who would not have had smallpox would have been $\frac{a^{(0)}}{n} - \frac{a^{(0)}}{mn}$. Now in order to know what number of persons of the first class is dead of other maladies, one will make the following proportion

$$a^{(0)} - \frac{a^{(0)}}{mn} : a^{(0)} - a^{(1)} - \frac{a^{(0)}}{mn} = a^{(0)} - \frac{a^{(0)}}{n} : \frac{a^{(0)} - \frac{a^{(0)}}{n}}{a^{(0)} - \frac{a^{(0)}}{mn}} (a^{(0)} - a^{(1)} - \frac{a^{(0)}}{mn}).$$

This proportion is reduced to say: The number of persons who would subsist if small-

[1] *Translator's note*: *Beyträge zum Gebrauche de Mathematik und deren Anwendung: Dritter Teil. Berlin: Buchhandlung der Realschule*, 1772. The section in question is "Unmerkungen über die Sterblichkeit, Todtenlisten, Geburten und Ehen," p. 476–599. See also the article "The mortality of smallpox in children," *History of actuarial science*, 1995, Vol. VIII, p. 39–54.

pox were the only mortal malady, is to the number of persons who are dead of other maladies, as the number of persons who would not have had smallpox if smallpox were the only mortal malady is to the number of these persons who are dead of other maladies. It is necessary to subtract this fourth number from the third, or the consequent from the antecedent, in order to have the number of living persons who have not had smallpox. This number is

$$= \frac{a^{(1)}\left(1-\frac{1}{n}\right)}{1-\frac{1}{mn}} = b^{(1)}.$$

One will object that one dies both from smallpox and from other maladies during the entire course of the interval calculated, & not at the end of this interval, as this proportion supposes it, but the interval being arbitrary, one can make it so small that the error is not sensible, as one will see below.

§ 7. In the second year there die under the assumption $a^{(1)}-a^{(2)}$ persons, there will be $\frac{b^{(1)}}{n}$ attacked by smallpox, & $\frac{b^{(1)}}{mn}$ who will die of it. There will die of it therefore $a^{(1)}-a^{(2)}\frac{b^{(1)}}{mn}$ of other maladies. If the other maladies had not carried a person away, the number of living persons who would not have had smallpox at the beginning of the third year, would have been $b^{(1)}-\frac{b^{(1)}}{n}$, & the number of persons who would have had it, would have been $a^{(1)}-b^{(1)}+\frac{b^{(1)}}{n}-\frac{b^{(1)}}{mn}$. I make, by repeating the reasoning exhibited for the first year, the following proportion

$$a^{(1)}-\frac{b^{(1)}}{mn} : a^{(1)}-a^{(2)}-\frac{b^{(1)}}{mn} = b^{(1)}-\frac{b^{(1)}}{n} : \frac{b^{(1)}-\frac{b^{(1)}}{n}}{a^{(1)}-\frac{b^{(1)}}{n}}\left(a^{(1)}-a^{(2)}-\frac{b^{(1)}}{mn}\right)$$

the fourth term of this proportion gives the number of persons exempted from smallpox who are dead of other maladies during the second year. Subtracting this fourth term from the third, I have for the number of persons who have not had smallpox at the beginning of the third year,

$$\frac{a^{(2)}\left(1-\frac{1}{n}\right)^2}{1-\frac{1}{m}+\frac{1}{m}\left(1-\frac{1}{n}\right)}.$$

§ 8. One will proceed precisely likewise for the third year & for each of the following years, this which will give the table joined here of which the law is manifest, so that one will conclude from it immediately the number of living persons who have not had smallpox at the beginning of the i^{th} year.

Years	Persons living exempted from smallpox.
0	$a^{(0)}$
1	$\dfrac{a^{(1)}\left(1-\frac{1}{n}\right)}{1-\frac{1}{m}+\frac{1}{m}\left(1-\frac{1}{n}\right)}$
2	$\dfrac{a^{(2)}\left(1-\frac{1}{n}\right)^2}{1-\frac{1}{m}+\frac{1}{m}\left(1-\frac{1}{n}\right)^2}$
3	$\dfrac{a^{(3)}\left(1-\frac{1}{n}\right)^3}{1-\frac{1}{m}+\frac{1}{m}\left(1-\frac{1}{n}\right)^3}$
4	$\dfrac{a^{(4)}\left(1-\frac{1}{n}\right)^4}{1-\frac{1}{m}+\frac{1}{m}\left(1-\frac{1}{n}\right)^4}$
$\dots$	$\dots$
i	$\dfrac{a^{(i)}\left(1-\frac{1}{n}\right)^i}{1-\frac{1}{m}+\frac{1}{m}\left(1-\frac{1}{n}\right)^i}$

Thus at the end of i years, or in general at the end of i intervals, the number of people who have not had smallpox will be

$$\frac{a^{(i)}\left(1-\frac{1}{n}\right)^i}{1-\frac{1}{m}+\frac{1}{m}\left(1-\frac{1}{n}\right)^i}=\frac{ma^{(i)}}{1+(m-1)\left(1-\frac{1}{n}\right)^{-i}}.$$

Supposing the intervals infinitely small & i infinite, one has

$$\left(1-\frac{1}{n}\right)^{-i}=1+\frac{i}{n}+\frac{i^2}{2n^2}+\frac{i^3}{2.3n^3}\text{ \&c.}=e^{\frac{i}{n}}.$$

The formula will become therefore

$$\frac{ma^{(i)}}{1+(m-1)e^{\frac{i}{n}}}.$$

We suppose now that these i intervals make x years, the formula becomes

$$\frac{ma^{(x)}}{1+(m-1)e^{\frac{s}{n}}}.$$

This is the formula which Mr. Bernoulli finds in the memoir cited. The calculation is, as one sees, quite simple, & requires not at all the integral calculus.

§ 9. It is evident that if i is infinite, one can multiply i & n by the same factor without that the value $\left(1-\frac{1}{n}\right)^{-i}$ changes, since the sequence contains only some multiples of $\frac{i}{n}$. And with supposing i infinite, it suffices to suppose it a little greater in order that

$\left(1-\frac{1}{n}\right)^{-i}$ is sensibly a constant value. This remark is essential for the usage that I myself propose to make of these formulas in the following of this memoir. I am going therefore to render it sensible by a detailed example. Let there be proposed to seek the number of living persons at the end of ten years & who have not had smallpox. According to the givens of Mr. Bernoulli one has $m = 8$, $n = 8$, $a^{(i)} = 661$, $i = 10$, (see the table that Mr. Bernoulli has calculated § 7. of his memoir). The formula is

$$\frac{8.661}{1+7\left(\frac{8}{7}\right)^{10}} = 192.$$

Here is the calculation of it

$$\begin{array}{rlrl}
\ln 8 = & 0.9030900 & \ln 661 = & 2.8202015 \\
\ln 7 = & \underline{0.8450980} & \ln 8 = & \underline{0.9030900} \\
\ln \dfrac{8}{7} = & 0.0579920 & \ln 8.661 = & 3.7232915 \\
10 \ln \dfrac{8}{7} = & 0.57799200 & \ln\left(1+7\left(\dfrac{8}{7}\right)^{10}\right) = & \underline{1.4410417} \\
\ln 7 = & \underline{0.8450980} & \ln \dfrac{8.661}{\left(1+7\left(\frac{8}{7}\right)^{10}\right)} = & 2.2822498 \\
\ln 7 + \ln\left(\dfrac{8}{7}\right)^{10} = & 1.4250180 & \dfrac{8.661}{\left(1+7\left(\frac{8}{7}\right)^{10}\right)} = & 192 \\
7\left(\dfrac{8}{7}\right)^{10} = & 26.6085 & & \\
1+7\left(\dfrac{8}{7}\right)^{10} = & 27.6085 & &
\end{array}$$

By conserving the same value $m = 8$ I multiply m & i by 64, this which gives $n = 512$, $i = 640$, next by 256, this which gives $n = 2048$, $i = 2560$, & I repeat the same calculation, as one see it in the following table:

$$
\begin{array}{rlrl}
\ln 512 = & 2.7092700 & \ln 2048 = & 3.3113300 \\
\ln 511 = & \underline{2.7084209} & \ln 2047 = & \underline{3.3111178} \\
\ln \dfrac{512}{511} = & 0.0008491 & \ln \dfrac{2048}{2047} = & 0.0002122 \\
640 \ln \dfrac{512}{511} = & 0.5434240 & 2560 \ln \dfrac{2048}{2047} = & 0.5432320 \\
\ln 7 = & \underline{0.8450980} & \ln 7 = & \underline{0.8450980} \\
\ln 7 \left(\dfrac{512}{511}\right)^{640} = & 1.3885220 & \ln 7 \left(\dfrac{2048}{2047}\right)^{2560} = & 1.3883300 \\
7 \left(\dfrac{512}{511}\right)^{640} = & 24.4637 & 7 \left(\dfrac{2048}{2047}\right)^{2560} = & 24.453 \\
1 + 7 \left(\dfrac{512}{511}\right)^{640} = & 25.4637 & 1 + 7 \left(\dfrac{2048}{2047}\right)^{2560} = & 25.453 \\
\ln 8.661 = & 3.7232915 & \ln 8.661 = & 3.7232915 \\
\ln \left(1 + 7 \left(\dfrac{512}{511}\right)^{640}\right) = & \underline{1.4059216} & \ln 1 + 7 \left(\dfrac{2048}{2047}\right)^{2560} = & \underline{1.4057390} \\
\ln \dfrac{8.661}{1 + 7 \left(\frac{512}{511}\right)^{640}} = & 2.3173699 & \ln \dfrac{8.661}{1 + 7 \left(\frac{2048}{2047}\right)^{2560}} = & 2.3175525 \\
\dfrac{8.661}{1 + 7 \left(\frac{512}{511}\right)^{640}} = & 207.67 & \dfrac{8.661}{1 + 7 \left(\frac{2048}{2047}\right)^{2560}} = & 207.75
\end{array}
$$

One sees that the result of these two calculations do not differ sensibly, & that the whole number which results from it is 208, as Mr. Bernoulli finds it. By multiplying thus n & i by 256, one will find by our formula all the numbers of the table of Mr. Bernoulli. This remark will be useful to us in the following.

§ 10. We suppose now that m & n instead of being constants, vary each year, so that at the end if the first year they are m' & n', at the end of the second, m'' & n'', at the end of the third m''' & n''', at the end of the i^{th} $m^{(i)}$ & $n^{(i)}$, we will have by repeating the same operations as above, for the number of living persons of each age who would not have had smallpox,

Years	Persons living exempted from smallpox.
0	$a^{(0)}$
1	$\dfrac{a^{(1)}\left(1-\frac{1}{n}\right)}{1-\frac{1}{mn}}$
2	$\dfrac{a^{(2)}\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n'}\right)}{1-\frac{1}{mn}-\frac{1}{m'n'}\left(1-\frac{1}{n}\right)}$
3	$\dfrac{a^{(3)}\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n'}\right)\left(1-\frac{1}{n''}\right)}{1-\frac{1}{mn}-\frac{1}{m'n'}\left(1-\frac{1}{n}\right)-\frac{1}{m''n''}\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n'}\right)}$
4	$\dfrac{a^{(4)}\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n'}\right)\left(1-\frac{1}{n''}\right)\left(1-\frac{1}{n'''}\right)}{1-\frac{1}{mn}-\frac{1}{m'n'}\left(1-\frac{1}{n}\right)-\frac{1}{m''n''}\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n'}\right)-\frac{1}{m'''n'''}\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n'}\right)\left(1-\frac{1}{n''}\right)}$
$\ldots$	$\ldots$
i	$\dfrac{a^{(i)}\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n'}\right)\cdots\left(1-\frac{1}{n^{(i-1)}}\right)}{1-\frac{1}{mn}-\frac{1}{m'n'}\left(1-\frac{1}{n}\right)-\frac{1}{m''n''}\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n'}\right)-\cdots\frac{1}{m^{(i-1)}n^{(i-1)}}\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n'}\right)\cdots\left(1-\frac{1}{n^{(i-2)}}\right)}$

If m changes alone & if n remains constant, one will have the following table:

Years	Persons living exempted from smallpox.
0	$a^{(0)}$
1	$\dfrac{a^{(1)}\left(1-\frac{1}{n}\right)}{1-\frac{1}{mn}}$
2	$\dfrac{a^{(2)}\left(1-\frac{1}{n}\right)^2}{1-\frac{1}{mn}-\frac{1}{m'n}\left(1-\frac{1}{n}\right)}$
3	$\dfrac{a^{(3)}\left(1-\frac{1}{n}\right)^3}{1-\frac{1}{mn}-\frac{1}{m'n}\left(1-\frac{1}{n}\right)-\frac{1}{m''n}\left(1-\frac{1}{n}\right)^2}$
4	$\dfrac{a^{(4)}\left(1-\frac{1}{n}\right)^4}{1-\frac{1}{mn}-\frac{1}{m'n}\left(1-\frac{1}{n}\right)-\frac{1}{m''n}\left(1-\frac{1}{n}\right)^2-\frac{1}{m'''n}\left(1-\frac{1}{n}\right)^3}$
$\ldots$	$\ldots$
i	$\dfrac{a^{(i)}\left(1-\frac{1}{n}\right)^i}{1-\frac{1}{mn}-\frac{1}{m'n}\left(1-\frac{1}{n}\right)-\frac{1}{m''n}\left(1-\frac{1}{n}\right)^2-\cdots\frac{1}{m^{(i-1)}n}\left(1-\frac{1}{n}\right)^{i-1}}$

§ 11. One can find different methods of approximation, according to the different values that one will assign to m, m', m'' &c, n, n', n'' &c. but the operation most commodious in practice seems to be to proceed by parts. Supposing, as above, the quantities m & n constants during the course of a year, but variables from one year to the other, & naming $b^{(1)}$, $b^{(2)}$, $b^{(3)}\ldots b^{(i)}$ the living persons who have not had smallpox at the end of $1,\ 2,\ 3\ldots i$ years, one will have

$$b^{(1)} = \frac{a^{(1)}\left(1-\frac{1}{n}\right)}{1-\frac{1}{m}+\frac{1}{m}\left(1-\frac{1}{n}\right)}, \qquad b^{(2)} = \frac{a^{(2)}b^{(1)}\left(1-\frac{1}{n'}\right)}{a^{(1)}-\frac{b^{(1)}}{m'}+\frac{b^{(1)}}{m'}\left(1-\frac{1}{n'}\right)},$$

$$b^{(3)} = \frac{a^{(3)}b^{(2)}\left(1-\frac{1}{n''}\right)}{a^{(2)}-\frac{b^{(2)}}{m''}+\frac{b^{(2)}}{m''}\left(1-\frac{1}{n''}\right)}, \qquad b^{(4)} = \frac{a^{(4)}b^{(3)}\left(1-\frac{1}{n'''}\right)}{a^{(3)}-\frac{b^{(3)}}{m'''}+\frac{b^{(3)}}{m'''}\left(1-\frac{1}{n'''}\right)},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \qquad b^{(i)} = \frac{a^{(i)}b^{(i-1)}\left(1-\frac{1}{n^{(i-1)}}\right)}{a^{(i-1)}-\frac{b^{(i-1)}}{m^{(i-1)}}+\frac{b^{(i-1)}}{m^{(i-1)}}\left(1-\frac{1}{n^{(i-1)}}\right)}.$$

One will determine successively each of these quantities, & in order to have a sufficient approximation, one will draw instead of $n,\ 256n,$ & instead of $1-\frac{1}{n},\ \left(1-\frac{1}{n}\right)^{256}$, as we have practiced above, Now, more simply again, when one will have determined $b^{(1)}$ by the formula

$$b^{(1)} = \frac{ma^{(1)}}{1+(m-1)\left(\frac{256n}{256n-1}\right)^{256}},$$

one will determine $b^{(2)}$ by the formula

$$b^{(2)} = \frac{m'a^{(2)}}{1+(m'-1)\left(\frac{256n'}{256n'-1}\right)^{2.256}},$$

$b^{(3)}$ by the formula

$$b^{(3)} = \frac{m''a^{(3)}}{1+(m''-1)\left(\frac{256n''}{256n''-1}\right)^{3.256}}\cdots$$

$b^{(i)}$ by the formula

$$b^{(i)} = \frac{m^{(i-1)}a^{(i)}}{1+(m^{(i-1)}-1)\left(\frac{256n^{(i-1)}}{256n^{(i-1)}-1}\right)^{256i}},$$

If one does not find that 256 gives a sufficient approximation, one could substitute a number as great as one would wish.

§ 12. If smallpox carried away all those that it attacked, one would have $m = m' =$ &c. $= 1$ & $b^{(1)} = a^{(1)}$, $b^{(2)} = a^{(2)}\ldots b^{(i)} = a^{(i)}$, that is to say the number of persons who have not had smallpox would be for each year equal to the number of the survivors, this which is evident, since there would never be among the living people who have had smallpox. If smallpox killed no person, one would have $\frac{1}{m} = \frac{1}{m'} =$ &c. $= 0$, or

$m = m' = \&c. = \infty$, &

$$b^{(1)} = a^{(1)}\left(1-\frac{1}{n}\right),$$
$$b^{(2)} = a^{(2)}\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n'}\right),$$
$$b^{(3)} = a^{(3)}\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n'}\right)\left(1-\frac{1}{n''}\right)$$
$$\ldots$$
$$b^{(i)} = a^{(i)}\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n'}\right)\cdots\left(1-\frac{1}{n^{(i-1)}}\right).$$

In general the value of

$$b^{(i)} = \frac{m^{(i-1)}a^{(i)}}{1+(m^{(i-1)}-1)\left(\frac{256n^{(i-1)}}{256n^{(i-1)}-1}\right)^{256i}}$$

decreases from $m = 1$, to $m = \infty$, for when $m = 1$, one has $b^{(i)} = a^{(i)}$, & when $m = \infty$, one has

$$b^{(i)} = \frac{a^{(i)}}{\frac{256n^{(n-1)}}{256n^{(i-1)}-1}}.$$

Thus $b^{(i)}$ is so much greater as m is smaller. To the contrary, $b^{(i)}$ is so much greater as n is greater, because if $n = \infty$, one has $b^{(i)} = a^{(i)}$, as it must be, since under this assumption no person contracts smallpox. If $n = 1$, one has $b^{(i)} = 0$, this which must be, since under this assumption everyone would have contracted smallpox from the first year.

§ 13. The advantage that our formulas have over those of Mr. Bernoulli would be imaginary, if one could not find any way to arrive, at the least nearly, to the values of m & n from year to year. We have already seen that if one knows the number of persons who die at each age of smallpox, as the table of mortality gives the number of persons who die at each age of some malady whatever it may be, & consequently the number of survivors at each age, the comparison of these values can give the fraction $\frac{1}{mn}$ for each year, that is to say the ratio of the number of persons who die of smallpox to the number of persons who have not had it. There is a question now to show how one can in each case arrive thence to knowledge of the values approached by m & n. I have found only two tables which might serve me for this object. The first that Mr. Lambert reports in the memoir cited, contains the number of persons dying at the Hague of smallpox during fifteen years with the designation of their age; the second is found in the German work that Mr. Möhsen has published under the title of Compilation[2] of experiences related to inoculation: it contains the number of persons dying at Berlin

[2]*Translator's note*: This is Johann Karl Wilhelm Möhsen, 1722-1795. The work in question appears to be *Sammlung merkwürdiger Erfahrungen, die den Werth und großen Nutzen der Pocken-Inoculation näher bestimmen können*. 1774/5. That is, "A Collection of strange experiences, that can determine nearer the Worth and great Benefits of Smallpox-Inoculation."

of smallpox during twelve years, from 1758 to 1774 with the designation of their age. This last table merits much more attention than the first, either because of its author of whom the merit is known, while I do not know the name of the one who has formed the table at the Hague, or because the number of deaths is nearly five times greater. In effect, there are dead at the Hague 1455 persons of smallpox during fifteen years, & there is dead at Berlin 6705 during seventeen years. Now one knows that in these sorts of calculations it is necessary to avoid with care taking small numbers for base, because the anomalies are too frequent, instead they compensate themselves with great numbers. Moreover, Mr. Möhsen has distinguished the years, instead in the table of the Hague they are confounded. Now in the table of Berlin, the greater part of the particular years, & especially those where the mortality has been considerable, follow a march analogous to that of the mean taken among the seventeen years, this which gives confidence in this march. Of the remaining, in the calculations of which I am going to take account, I have paid attention only to the mean among the seventeen years, because the numbers of deaths at each particular age vary too much according as the epidemic has reigned or not. I would have likewise desired to be able to take a mean after a more considerable number of years, but I have not been able. The table of Mr. Möhsen has presented me a difficulty, it is that the number of deaths are marked from year to year only for the first five years of life, these numbers are marked next only from five to five years, this which has engaged me to interpolate in order to make follow with these numbers a march nearly regular. Besides, I give here only for a very crude test, uniquely destined to indicate the part that one could deduce from more exact & more complete tables, & the interesting consequences which would result from it. I will not neglect entirely the observations of the Hague; although their results differ in many regards from the results of the observations of Berlin, there are however among them some analogies that it is good to remark.

§ 14. As my principal end is to find the differences which exist between the our method & that of Mr. Bernoulli, I will conserve all the assumptions of this grand geometer which do not deviate much from experience. Thus I will suppose with Mr. Bernoulli that $\frac{1}{13}$ of those who die of smallpox, although the ratio which results from these seventeen years is a little greater, because one finds that there is dead a common year in Berlin 4772 persons, & that there are dead of them 395 of smallpox, this which gives around $\frac{1}{12\frac{1}{10}}$ instead of $\frac{1}{13}$. But as this change would have rendered very difficult the comparison that I have in view, & that besides among the seventeen years there have been many of them who have been quite murdered, I myself am held to $\frac{1}{13}$. Besides this difference is of no importance for the nature of the consequences that I myself propose to deduce. I will conserve also the method of Mr. Bernoulli to compare the number of deaths of each year not to the number of persons existing at the beginning of the year, but to the number of persons existing in the middle of the year, that is to say to the mean number between the numbers which correspond to two consecutive years. This requires, to say truly, a tentativeness in my method, because the second number is not known; but it is easy to determine nearly according to those of Mr. Bernoulli, & if it is ill-determined, the sequence of the calculation shows it. The calculation of the first year is more vague, because Mr. Bernoulli has determined the number of births according to an assumption, & not according to the table of Halley which begins only at the first year.

§ 15. The total number of deaths is according to Mr. Möhsen 6705, & the one of the deaths of the first year is 1790. The total number of deaths according to Mr. Bernoulli being 100, I make the proportion $100 : x = 6705 : 1790$, & I find $x = 26.7$. Mr. Bernoulli finds $x = 17.1$, thus the deaths of the first year are much more numerous than it should be according to the assumptions of Mr. Bernoulli. I suppose $m = 6$, & I multiply by 6 the number of deaths, this which gives me 160, I subtract this number from 1300 & I have 1140, I diminish this number in the proportion of 1300 to 1000 & I have 877. I take the mean between 1300 & 877 & I have 1088; I divide this number by 26.7, & I have 41. As one must have $mn = 41$, I make $n = 7$, & I calculate the formula

$$\frac{6.1000}{1+5\left(\frac{1792}{1791}\right)^{256}} = 887.$$

This number being ⟨added to 1300, then⟩[3] divided by $\langle 2.\rangle 42$ gives 26.4, this which differs little from 26.7, I keep myself to it there, & I pass to the second year. The number of deaths being 1416 according to Mr. Möhsen, I make the proportion $100 : x = 6705 : 1416$, & I find $x = 21.1$. I leave $m = 6$, this which gives me 127 for the number of deaths, I subtract this number from 887 & I have 760, I diminish this number in the ratio of 1000 to 855 & I have 650. I take the mean 887 & 650, this which gives me 768.5, I divide this number by 21.1, & I have 36, therefore $n = 6$; I calculate the formula

$$\frac{6.855}{1+5\left(\frac{1536}{1535}\right)^{2.256}} = 643,$$

this number ⟨added to 887, then⟩ divided by $\langle 2.\rangle 36$ gives 21.2 this which accords quite well with the assumption. For the third year, the number of deaths being 1113, I make the proportion $100 : x = 6705 : 1116$, & I find $x = 16.6$. I leave $m = 6$, this which gives 98 for the number of deaths, I subtract this number from 643 & I have 545, I diminish this number in the ratio of 855 to 798 & I have 509. I take the mean between 509 & 643 & I have 576. I divide this number by 16.6 & I have $n = 5.7$. I calculate the formula

$$\frac{6.798}{1+5\left(\frac{1459.2}{1458.2}\right)^{3.256}} = 506,$$

this number being ⟨added to 643, then⟩ divided by $\langle 2.\rangle 34.7$ gives 16.6, as this must be. For the fourth year, the number of deaths being 1001, I make the proportion $100 : x = 6705 : 1001$ & I have $x = 15$, the assumption of $m = 6$ gives me a final result which is separated from the primitive assumption, I make therefore $m = 5.5$ this which give 82 for the number of deaths, I subtract this number from 506 & I have 424, I diminish this number in the ratio of 760 to 732 & I have 408. I take the mean between 506 & 408 & I have 456.5: this number divided by 15 gives 30.4, this last number divided by

[3] *Translator's note*: Here and throughout this section, there were omissions in the impression. The corrections, as additions to the text, have been placed between the symbols ⟨ and ⟩. See the *Éclaircissement* in the Volume for 1796 of this journal.

5.5 gives $n = 5.5$; I calculate the formula

$$\frac{5.5.732}{1 + 4.5\left(\frac{1408}{1407}\right)^{4.256}} = 403,$$

this number $\langle$added to 506, then$\rangle$ divided by $\langle 2.\rangle 30.4$ gives 15, as this must be. For the fifty year, the number of deaths being 556 I make the proportion $100 : x = 6705 :$ 556, I find $x = 8.3$. In order that the result of the calculation be in accord with the assumption, I am obliged to increase the value of m; I make therefore $m = 8$, & I have the number of deaths $= 66$, this which subtracted from 403 gives 337. I diminish this number in the ratio of 732 to 710 & I have 327, I take the mean between 403 & 327 & I have 365, this which divided by 8.3 gives 44, this which divided by 8, gives $n = 5.5$. I calculate the formula

$$\frac{8.710}{1 + 7\left(\frac{1408}{1407}\right)^{5.256}} = 319,$$

this number $\langle$added to 403, then$\rangle$ divided by $\langle 2.\rangle 44$, gives 8.2, as this must be. One sees that all the artifice of this tentativeness or of this method of false position is to suppose a certain value of m, to calculate nearly under this supposition the value of n, & to seek next directly according to these values of m & n by the formula reported above the number of living persons who have not had smallpox, & to deduce from this number by means of the value mn the number of deaths, which number must accord itself with observation. One varies the value of m until one finds this accord, this which is never too long, because one knows how the result of the formula varies according to the variations of m & of n. This method, as imperfect as it is, has seemed to me to lead more directly to the end than some more scholarly & more complicated methods, of which I myself abstain to make mention, at least until one has some more complete tables of observations. It would be useless to report here in detail the calculation of the following years, since the march is always the same, so much the more as having the number of deaths only from 5 to 5 years, I have deduced only by interpolation the number of deaths for each year. I will content myself therefore to give a table of values of n & m that I have obtained for each year by the preceding method.

Years	n	m	Years	n	m
1	7	6	19	5.8	60
2	6	6	20	5.8	74
3	5.7	6	21	5.8	74
4	5.5	5.5	22	5.8	74
5	5.5	8	23	5.8	74
6	5.5	11	24	5.8	93
7	5.5	14	25	5.8	133
8	5.5	17	26	5.8	133
9	5.5	30	27	5.8	112
10	5.5	72	28	5.8	92
11	5.5	120	29	5.8	78
12	5.5	112	30	5.8	63
13	5.5	100	31	5.8	52
14	5.65	90	32	5.8	44
15	5.7	80	33	5.8	37
16	5.7	74	34	5.8	31
17	5.7	67	35	5.8	26
18	5.7	63			

Years	m
from 0 to 5 years incl.	6.3
6–10	29
11–15	100
16–20	65
21–25	90
26–35	67

§ 16. I have calculated this table, by interpolating for each year, only until 25 years, from 26 to 35 I have supposed the number of deaths the same for each year, the numbers were too small in order that it be worth the pain to interpolate, & my end was only to obtain a mean value for this interval of ten years. I have not been able to push the calculation to the end of 35 years, because the numbers were too small. There results from this table, 1° that the value of n does not vary sensibly, & is found contained within the narrow limits of $5\frac{1}{2}$ & 6; the first year alone gives $n = 7$, but the calculation of this year is the least exact, as I have observed above. Thus according to these observations, it is true to say that for each age the ratio of the number of those who contract smallpox to the number of those who have not contracted it, is nearly constant, & in this the supposition of Mr. Bernoulli deviates itself little from the truth; but he made the danger to contract smallpox smaller than it is according to these observations, because he made it $\frac{1}{8}$ & because it is here a little greater than $\frac{1}{6}$. 2° that the value of m varies considerably, so that the danger to die of smallpox when one has contracted it is around $\frac{1}{6}$ during the first three years of life, that this danger attains the *maximum* at the fourth year, where it is $\frac{1}{5.5}$; that it diminishes rapidly during the following six years, & at the eleventh year attains a kind of *minimum*, it is then only $\frac{1}{120}$, this danger increases anew from the twentieth year where it is $\frac{1}{60}$, next it rediminishes to the twenty-fifth year where it is $\frac{1}{133}$, that is to say smaller than at the eleventh year. If one wishes to be content to take the mean danger from five to five years, one will find it in the second table of § 15. According to this table the danger is greatest in the first five years of life, it is smallest between 10 & 15 years, it is greater between 15 & 20 years than in the five years which precede & in the five which follow, because it is $\frac{1}{65}$ between 15 & 20 years, & only $\frac{1}{100}$ between 10 & 15 & $\frac{1}{90}$ between 20 & 25. This result appears rather natural, because the age from 15 to 20 years is a critical age, especially for the women, this which must render smallpox more dangerous. Finally from 26 to 35 years the danger recommences to increase, & a light test of calculation on the years following gave me a rapid increase, but not wishing at all to admit of hypotheses I am myself abstained from determining this increase. One sees therefore that on this point Mr. Bernoulli has made an inadmissible assumption, & that the danger to die of smallpox when one has contracted it, is not at all the same at each age.

§ 17. I am going now to the observations of the Hague of which I have calculated only the first 12 years. In the following the numbers are so small that there results from it great anomalies, it would be necessary to interpolate, & the same interpolation would have been very little certain because of the smallness of the numbers.

Years	n	m	Years	n	m
1	5.5	16.5	7	7	5.5
2	8	8.1	8	7	7
3	8	6.4	9	7	8
4	7.5	4.6	10	7	16
5	7	5.5	11	7	29
6	7	6.2	12	7	19

The results differ in some points from the results of the observations of Berlin, but in some other points they have with those a marked analogy. The value of n does not vary much, no more than at Berlin, with the exception of the first year, but this value is greater than at Berlin, so that at the Hague the danger to contract smallpox at each age would be around $\frac{1}{7}$, instead that it is at Berlin a little greater than $\frac{1}{6}$. As for the value of m, or to the danger to die of smallpox when one has contracted it, it is much smaller the first year at the Hague than at Berlin, however this danger has a maximum in the fourth year, as at Berlin, & a minimum at the eleventh year as at Berlin, (this which is rather remarkable,) although it diminishes much less rapidly at the Hague than at Berlin. But seeing the small number of observations, it is necessary to pay more attention to the law which the number follow than to their absolute value, & this analogy in the law, in spite of the particular varieties, merits without doubt to be observed. Besides there can be some differences in the manner smallpox to act, which depends on the climate & on the situation of the country. And the process that I just exposed appears proper to put them in hand.

§ 18. Since the value of n varies too little, one could suppose it constant, & to calculate the values of m according to the formulas given at the end of §10. But as long as one will not have collected a greater number of observations, it will not be worth scarcely the pain to be delivered to these calculations which besides are susceptible of interesting diverse applications. It is therefore to desire that one can procure oneself some lists of persons dead of smallpox with the designation of the age for a long sequence of years, & that in the cities a little large where the particular anomalies can not prevent that one distinguish the general law. One would have thence a very simple means to appreciate the effects of inoculation. I suppose for example that one made an analogous calculation to the one that I am going to make for the 50 years which have preceded the introduction of inoculation, one would know for each age the danger to contract smallpox & the danger to die of it independently of inoculation. Making next a similar calculation for the years which have followed the introduction of inoculation, one would see that which one has won in these two regards, & one could decide thence not only the question of mortality, but yet that of the danger of contracting smallpox, that inoculation can be augmented in perpetuating the epidemic. One would see if the mean risk to contract smallpox at each age has increased by inoculation, & if the number of persons who die without having had smallpox has diminished thence. As for the question often considered, at what age it is necessary to inoculate, the thing is more complicated than it seems at first glance. By supposing, that which is rather natural, that the danger of inoculation follows the same law as the one of natural smallpox, it is clear that inoculation would be much more dangerous in the first five years of life than in the following, but by differing this operation one courts an eminent risk of allowing oneself to prevent it by natural smallpox, & that in the age where this malady is most deadly. The combination of these two kinds of risks is delicate, it requires some precautions of which the detail would carry me away too far. It suffices for me to have indicated also briefly that it has been possible myself, how one could pour out more of day on an important question. The application that I have tested, is, I repeat it, only a test proper to serve as example, & not to make known the true laws of nature.